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A Geometric Approach to Minimum-Time Eigen-Axis Repointing Maneuvers in Spacecraft with Pyramidal Reaction Wheels

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ABSTRACT

In this paper, a geometric approach is proposed for the design of minimum-time eigen-axis repointing maneuvers of symmetric spacecraft equipped with four reaction wheels in a pyramidal configuration. First, by analyzing the geometric distribution of the available control torque, an algorithm is developed to compute the maximum achievable torque along an arbitrary direction. Based on this maximum torque, the constant control coefficients required for implementing a bang-bang control law in the eigen-axis maneuver are determined. Analytical and numerical results demonstrate that, due to the non-uniform torque distribution inherent in the pyramidal configuration, selecting an eigen-axis perpendicular to the plane defined by the initial and final pointing directions does not necessarily yield the minimum maneuver time. Instead, the optimal eigen-axis is located at an angle smaller than 90 degrees with respect to this plane. Furthermore, the torque distribution and maneuver performance are investigated under the condition of failure of one reaction wheel and compared with the conventional three orthogonal reaction wheel configuration. Numerical simulations show that the pyramidal configuration provides superior maneuver performance in terms of achievable torque and maneuver time, even in the presence of a single wheel failure.

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1. INTRODUCTION

Solving the minimum-time attitude maneuver problem remains a fundamental challenge in spacecraft attitude control. While Control Moment Gyros (CMGs) [1] are frequently employed to enhance agility and reduce maneuver duration, reaction wheels remain a preferred choice for high-precision missions, such as Earth observation and imaging, due to their superior accuracy. A robust strategy to optimize performance is the utilization of a four-wheel pyramidal configuration. This setup not only improves torque distribution compared to traditional three-orthogonal wheel arrangements but also provides essential redundancy, ensuring mission continuity in the event of a single-wheel failure. Consequently, the pyramidal configuration is a highly efficient and reliable solution for both attitude control and time-optimal maneuvering

Yoon et al. [2] proposed methods for enhancing the agility of spacecraft equipped with four reaction wheels in a pyramidal configuration, which, in addition to simplifying computations, significantly improve spacecraft maneuverability. Kasiri et al. [3] investigated the optimal wheel skew angle to reduce angular momentum during maneuvers, thereby preventing wheel saturation and optimizing fuel consumption; they also introduced a new method for determining these optimal angles based on the geometry of torque and momentum distribution. Karpenko and King [4] presented a method for the geometric distribution of agility using reaction wheels, enabling the design of optimal wheel angles. Sumithra and Joglekar [5] proposed a maneuver approach for high-precision spacecraft that maximizes angular acceleration along the eigen-axis, thus reducing maneuver time. Kron et al. [6] developed algorithms for torque and angular momentum distribution to optimize maneuver performance; these algorithms were subsequently applied to the PROBA-3 mission, where their operational effectiveness was confirmed. Yoon et al. [7] presented an algorithm to compute torque and angular momentum distribution in reaction wheel arrays, as well as the allocation of torque or momentum among the wheels, leading to more efficient spacecraft design. Lee [8] introduced a novel method for determining the optimal momentum bias in long-duration maneuvers using a numerical approach that incorporates an optimal computational scheme and gain tables for rapid solution access. Finally, Sugita [9] introduced a torque distribution algorithm that, by accounting for wheel torque and angular momentum, improves spatial agility and reduces energy consumption. Ashok Kumar et al. [10] also

presented an algorithm for optimal torque distribution that increases spacecraft maneuverability and agility using reaction wheels; this algorithm is based on the null motion of the torque-alignment matrix, which allows torque to be distributed equally among the reaction wheels, increases spacecraft maneuverability, and reduces the risk of wheel saturation. Ismail and Varatharajoo [11] examined various reaction wheel configurations for a three-axis attitude control subsystem and showed that the pyramidal configuration is the best option for minimum energy consumption. Newton et al. [12] compared the performance of different control laws for CubeSats with various wheel configurations in the presence of faults and found that the sliding-mode controller outperforms the feedback-linearization controller; this finding is of great importance in ensuring robustness and fault tolerance in CubeSat attitude control systems. In addition, Markley et al. [13] presented algorithms for determining the maximum torque and momentum distribution for any reaction wheel configuration that, using a pseudo-inverse method, improve the agility and maneuverability of the spacecraft in configurations with more than three wheels.

In this research, the problem of the minimum-time eigen-axis maneuver is investigated for symmetric spacecraft equipped with four reaction wheels in a pyramidal configuration. First, an algorithm is presented that is able to compute the maximum torque applicable in an arbitrary direction by these four wheels and to determine the control coefficient of each wheel for implementing the bang-bang control law in the eigen-axis maneuver. The main difference between this study and previous work [13] is that here the torque envelope along the eigen-axis is determined explicitly, and the control coefficients are obtained specifically for the minimum-time repointing maneuver. The results show that, in the pyramidal configuration, the maximum torque distribution forms a regular dodecagon. The eigen-axis can be any arbitrary vector lying in the plane connecting the initial and final pointing directions; however, the vector with the smallest rotation angle does not necessarily minimize the maneuver time. Therefore, before executing the minimum-time maneuver, the optimal eigen-axis is first determined, and then, once it and the maximum torque along it are known, the bang-bang control law is applied to perform the maneuver. Furthermore, under the condition that one of the reaction wheels fails, the torque distribution takes the shape of an oblique prism with a rhombic cross-section; compared with the three-orthogonal-wheel configuration, the maneuverability is reduced and the control becomes more limited, yet the spacecraft remains able to continue

its mission.

After reviewing previous studies and introducing the contributions of this research, the subsequent sections of the paper present the dynamics and kinematics equations of the spacecraft with four reaction wheels in the pyramidal configuration, providing the basis for the analysis and design of the algorithms. Next, the algorithm for computing the maximum torque is described, and the method for determining the control coefficient of each wheel for executing the maneuver is presented. Then, the minimum-time Eigen-axis repointing problem for the symmetric spacecraft is examined, and the bang-bang control law for executing these maneuvers is presented. Finally, the simulation and test results for three different attitude-maneuver cases demonstrate the effectiveness of the proposed method and the effects of the pyramidal configuration and torque distribution.

2. ROTATIONAL EQUATIONS OF MOTION OF THE SPACECRAFT

The Euler dynamic equations of rotational motion for a rigid body in a body frame whose origin is located at the body's center of mass are given as follows [14]:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega} = \mathbf{u} \quad (1)$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$ is the angular velocity vector expressed in the body frame, $\mathbf{u} = [u_1, u_2, u_3]^T$ is the control torque vector, and $\mathbf{I} \in \mathbb{R}^{3 \times 3}$ is the inertia matrix in the body frame.

The kinematic equations, using the modified Rodrigues parameters to describe the attitude of a rigid body, are given as follows:

$$\dot{\boldsymbol{\sigma}} = \mathbf{B}\boldsymbol{\omega} \quad (2)$$

where

$$\mathbf{B} = \frac{1}{4}[(1 - \boldsymbol{\sigma}^T \boldsymbol{\sigma}) \mathbf{I}_3 + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T + 2\boldsymbol{\sigma}^\times] \quad (3)$$

and $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3]^T$ is the modified Rodrigues parameter vector, defined as

$$\boldsymbol{\sigma} = \mathbf{e} \tan\left(\frac{\phi}{4}\right) \quad (4)$$

where $\mathbf{e} = [e_1, e_2, e_3]^T$ is the eigen-axis vector and ϕ is the rotation angle about it.

3. ALGORITHM FOR COMPUTING THE MAXIMUM TORQUE AND CONTROL COEFFICIENTS OF THE REACTION WHEELS

In this research, the maximum torque along an arbitrary direction is defined as the largest torque that can be produced along an arbitrary unit vector $\mathbf{n}$, such that constraints imposed on the reaction wheels, the torque limit $u_{\max}$ of each wheel, are respected. In other words, the goal is to determine the vector of control coefficients $\mathbf{c}$ of the reaction wheels such that the component of the produced torque vector $\mathbf{u}$ along the desired direction $\mathbf{n}$ is maximized, while the produced torque vector $\mathbf{u}$ remains within the set of torques achievable by the system.

In this section, the computation of the maximum torque $u_{e\max}$ and the control coefficient vector $\mathbf{c}$ of each reaction wheel is examined for three different cases:

- Four reaction wheels in a pyramidal configuration
- Three reaction wheels in a pyramidal configuration
- Three orthogonal reaction wheels

First, the algorithm for computing the maximum torque and the control coefficients is presented, and then the geometric distribution of the torque vector $\mathbf{u}$ in three-dimensional space is displayed.

3.1. Four Reaction Wheels in a Pyramidal Configuration

In the pyramidal configuration, as shown in Fig. 1, the axis of each reaction wheel makes a skew angle β with the xoy plane.

For an arbitrary attitude maneuver, consider the eigen-axis vector $\mathbf{e} = [e_1, e_2, e_3]^T$. The goal is to compute the maximum torque that can be applied along the eigen-axis by the four reaction wheels. For this purpose:

- The control space produced by vectors u_1 and u_3 is denoted by p .
- The control space produced by vectors u_2 and u_4 is denoted by q .

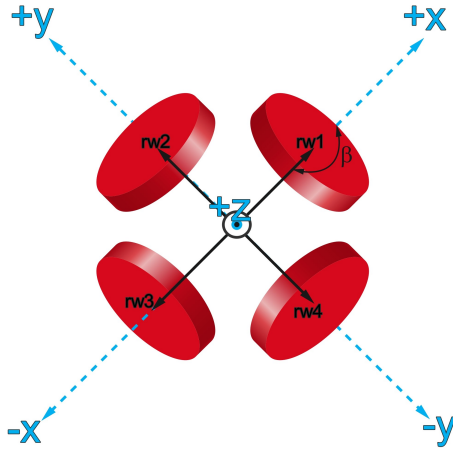


Fig. 1. Pyramidal configuration: as shown in the figure, each wheel is positioned at a skew angle β relative to the xy plane.

Using these spaces, the maximum torque along the eigen-axis is computed and the control coefficients of the corresponding reaction wheels are determined.

$$p: \frac{|x|}{2\cos\beta} + \frac{|z|}{2\sin\beta} \leq 1 \quad (5)$$

$$q: \frac{|y|}{2\cos\beta} + \frac{|z|}{2\sin\beta} \leq 1 \quad (6)$$

where β is the skew angle of the reaction wheels with respect to the xy plane. Fig. 2 shows the planes p and q .

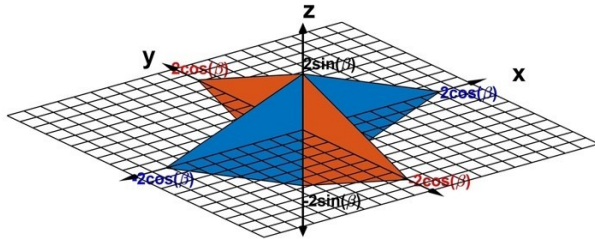


Fig. 2. Torque from reaction wheels 1 and 3 (blue) and torque from reaction wheels 2 and 4 (red).

In the general case, it is assumed that the vector $\mathbf{e}$ does not lie in either of the planes p or q . Consider an arbitrary plane r that passes through the vector $\mathbf{e}$. The intersection of this plane with the planes p and q produces two half-lines that meet at the origin. The normal vector of plane r is denoted by $\mathbf{n} = [n_1, n_2, n_3]^T$, which must satisfy the following conditions:

$$\mathbf{n} \perp \mathbf{e}, \quad \|\mathbf{n}\| = 1 \quad (7)$$

Eq. (7) consists of two equations with three unknowns; hence, the vector $\mathbf{n}$ is not uniquely determined. However, since the goal is to compute the maximum torque along the direction of $\mathbf{e}$, the vector $\mathbf{n}$ will be uniquely determined by imposing this condition, as described below.

The equation of the plane passing through the vector $\mathbf{e}$ is obtained as follows:

$$n_1x + n_2y + n_3z = 0 \quad (8)$$

The equation of the intersection of planes r and p is obtained by setting $y = 0$ in Eq. (8):

$$n_1x + n_3z = 0 \quad (9)$$

Eq. (9) specifies a line that passes through the origin and lies in plane p . This line intersects the rhombic boundary of the torque distribution at two points. For this equation, three different cases can be considered:

Case 1: If $n_1 \neq 0$, $n_3 \neq 0$. The system of equations formed by Eqs. (5) and (9) determines two points. The vector $\mathbf{k}_p$ is defined as the vector starting at the origin and ending at the intersection point. The magnitude and direction of this vector are obtained from the solution of the system of equations as follows:

$$k_p = |z| \sqrt{1 + \left(\frac{n_3}{n_1}\right)^2} = \frac{\sqrt{1 + \left(\frac{n_3}{n_1}\right)^2}}{\frac{1}{2\sin\beta} + \frac{\left|\frac{n_3}{n_1}\right|}{2\cos\beta}} \mathbf{e}_p = \pm \frac{\left[\frac{1}{n_1}, 0, -\frac{1}{n_3}\right]^T}{\left\|\left[\frac{1}{n_1}, 0, -\frac{1}{n_3}\right]^T\right\|} \quad (10)$$

Case 2: If $n_1 = 0$, $n_3 \neq 0$, the magnitude and direction of the vector $\mathbf{k}_p$ are as follows:

$$k_p = 2\cos\beta, \quad \mathbf{e}_p = [\pm 1, 0, 0]^T \quad (11)$$

Case 3: If $n_1 \neq 0$, $n_3 = 0$, the magnitude and direction of the vector $\mathbf{k}_p$ are obtained as follows:

$$k_p = 2\sin\beta, \quad \mathbf{e}_p = [0, 0, \pm 1]^T \quad (12)$$

The angle α_p is defined as the acute angle between $\mathbf{e}_p$ and $\mathbf{e}$, computed from the following relation:

$$\alpha_p = \cos^{-1}(|\mathbf{e}_p^T \mathbf{e}|) \quad (13)$$

Similarly, the values k_q , $\mathbf{e}_q$, and α_q are computed for plane q .

In special cases, the eigen-axis vector $\mathbf{e}$ may lie in one of the planes p or q . As an example, consider the case in

which the vector $\mathbf{e}$ lies in plane p . In this case, the angles α_p and α_q are defined as follows:

$$\alpha_p = \cos^{-1} |e_1|, \quad \alpha_q = \frac{\pi}{2} - \alpha_p \quad (14)$$

and the vectors $\mathbf{k}_p$ and $\mathbf{k}_q$ are determined from the following relations:

- a) If $\alpha_p < \beta$, the magnitude and direction of $\mathbf{k}_p$ and $\mathbf{k}_q$ are given by:

$$k_p = 2 \cos \beta, \quad k_q = 2 \cos \beta \tan \alpha_p$$

$$\mathbf{e}_p = [\text{sgn}(e_1), 0, 0]^T, \quad \mathbf{e}_q = [0, 0, \text{sgn}(e_3)]^T \quad (15)$$

- b) If $\alpha_p > \beta$, the magnitude and direction of $\mathbf{k}_p$ and $\mathbf{k}_q$ are:

$$k_q = 2 \sin \beta, \quad k_p = 2 \sin \beta \cot \alpha_p$$

$$\mathbf{e}_q = [\text{sgn}(e_1), 0, 0]^T, \quad \mathbf{e}_p = [0, 0, \text{sgn}(e_3)]^T \quad (16)$$

Having determined the vectors $\mathbf{k}_p$ and $\mathbf{k}_q$ in planes p and q , the next step is to ensure their resultant aligns with the eigen-axis $\mathbf{e}$. To achieve this, one vector must be scaled based on the limiting torque condition. Identifying the limiting vector is performed as follows:

$$\text{If } \frac{k_q}{k_p} > \frac{\sin \alpha_p}{\sin \alpha_q} = m \Rightarrow \begin{cases} \bar{k}_p = k_p \\ \bar{k}_q = mk_p \end{cases} \quad (17)$$

$$\text{If } \frac{k_q}{k_p} < m \Rightarrow \begin{cases} \bar{k}_p = k_q/m \\ \bar{k}_q = k_q \end{cases}$$

The maximum torque applicable along the eigen-axis is subsequently computed as:

$$u_{e\max} = \sqrt{\bar{k}_p^2 + \bar{k}_q^2 + 2\bar{k}_p\bar{k}_q \cos(\alpha_p + \alpha_q)} \quad (18)$$

With the exception of the special cases where the eigen-axis $\mathbf{e}$ resides within plane p or q , Eq. (18) is a function of the vector $\mathbf{n}$. Consequently, to complete the optimization, vector $\mathbf{n}$ must be determined such that it satisfies the constraints of Eq. (7) while maximizing the value in Eq. (18).

Finally, the control coefficients c_1, c_3 and c_2, c_4 are derived from $\mathbf{e}_p, \bar{k}_p$ and $\mathbf{e}_q, \bar{k}_q$, respectively, such that the following relations hold:

$$c_1 u_m \begin{bmatrix} \cos \beta \\ 0 \\ \sin \beta \end{bmatrix} + c_3 u_m \begin{bmatrix} -\cos \beta \\ 0 \\ \sin \beta \end{bmatrix} = \bar{k}_p \mathbf{e}_p \quad (19)$$

$$c_2 u_m \begin{bmatrix} 0 \\ \cos \beta \\ \sin \beta \end{bmatrix} + c_4 u_m \begin{bmatrix} 0 \\ -\cos \beta \\ \sin \beta \end{bmatrix} = \bar{k}_q \mathbf{e}_q$$

These control coefficients, c_i ($i = 1, 2, 3, 4$), represent the gain for each reaction wheel and determine the fraction of individual torque capacity required to execute the time-optimal maneuver.

The steps for computing the maximum torque along the eigen-axis and determining the control coefficients of the reaction wheels are summarized in Pseudocode 1.

Pseudocode 1: Algorithm for computing maximum eigen-axis torque in a pyramidal configuration

Inputs: eigen-axis vector $\mathbf{e}$, wheel skew angle β , maximum wheel torque u_m

Outputs: Maximum torque along the eigen-axis $u_{e\max}$, control coefficients c_1, c_2, c_3, c_4

1. Define control planes $p(u_1, u_3)$ and $q(u_2, u_4)$.
2. Define plane r such that it contains vector $\mathbf{e}$.
3. **Case I: $\mathbf{e}$ does not lie in plane p or q :**
 - 3.1. Compute the intersection line of planes r and p ; determine $k_p, \mathbf{e}_p$, and α_p using Eq. (10).
 - 3.2. Compute the intersection line of planes r and q ; determine $k_q, \mathbf{e}_q$, and α_q using corresponding relations.
 - 3.3. Determine the limiting vector by comparing k_p and k_q via Eq. (17); compute scaled values $\bar{k}_p$ and $\bar{k}_q$.
 - 3.4. Formulate $u_{e\max}$ as a function of vector $\mathbf{n}$ using Eq. (18).
 - 3.5. Determine optimal $\mathbf{n}$ satisfying Eq. (7) that maximizes $u_{e\max}$.
4. **Case II: $\mathbf{e}$ lies in plane p or q :**
 - 4.1. Calculate angles α_p and α_q .
 - 4.2. Determine $k_p, k_q, \mathbf{e}_p$, and $\mathbf{e}_q$ by comparing α_p and α_q with β (see Eqs. (15)-(16)).
 - 4.3. Identify the limiting vector via Eq. (17) and compute $\bar{k}_p$ and $\bar{k}_q$.
5. Solve Eq. (19) to determine control coefficients c_1, c_2, c_3 , and c_4 .
6. Output $u_{e\max}$ and $\{c_1, c_2, c_3, c_4\}$.

3. 2. Three Reaction Wheels in a Pyramidal Configuration (Single Wheel Failure)

In this scenario, a single actuator failure is considered where the fourth reaction wheel (u_4) becomes inoperative. Consequently, the achievable torque set is constrained, and its boundary in three-dimensional space deforms due to the loss of this actuator's contribution. The achievable region is therefore governed solely by the three functional wheels, resulting in a reduced torque capability compared to the nominal four-wheel configuration.

Fig. 4 illustrates the three-dimensional torque capability envelope under this degraded condition. The computation of the maximum torque and the associated control coefficients along the eigen-axis proceeds analogously to the procedure described in previous Section. However, the control space in plane $q = \text{span}\{u_2\}$ degenerates from a planar rhombus into a one-dimensional line segment of length $2u_m$, driven exclusively by the second wheel. Accordingly, the maximum applicable torque and the corresponding unit direction vector in plane q are given by:

$$k_q = u_m, \quad \mathbf{e}_q = \pm[0, \cos \beta, \sin \beta]^T \quad (20)$$

With $\mathbf{k}_p$ and the eigen-axis vector $\mathbf{e}$ fully determined, the vector $\mathbf{k}_q$ is uniquely defined due to the coplanarity of $\mathbf{k}_p$, $\mathbf{k}_q$, and $\mathbf{e}$. The subsequent steps of the algorithm including constraint evaluation and determination of the control coefficients proceed identically to the formulation presented in the previous section.

3.3. Three Orthogonal Reaction Wheels

For the orthogonal three-reaction-wheel configuration, the maximum eigen-axis torque and the corresponding normalized control coefficients are derived directly as:

$$u_{e\max} = \min \left\{ \frac{u_m}{|e_1|}, \frac{u_m}{|e_2|}, \frac{u_m}{|e_3|} \right\}, \quad (21)$$

$$c_1 = e_1, \quad c_2 = e_2, \quad c_3 = e_3$$

Once the control coefficients c_i are determined for either the four-reaction-wheel configuration ($i = 1, 2, 3, 4$) or the three-reaction-wheel configurations ($i = 1, 2, 3$), the maximum control torque applicable by each reaction wheel is obtained from:

$$u_{i(\max)} = c_i u_m \quad (22)$$

Figures 3–5 illustrate the geometric distribution of the maximum torque for the three considered configurations.

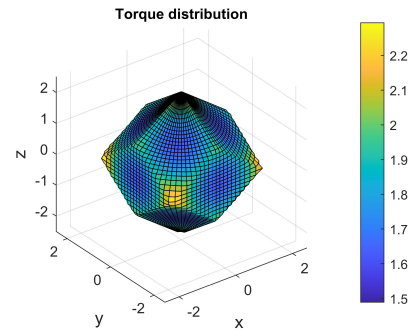


Fig. 3. Maximum torque distribution in the pyramid configuration with a skew angle of 35 degrees of the reaction wheels.

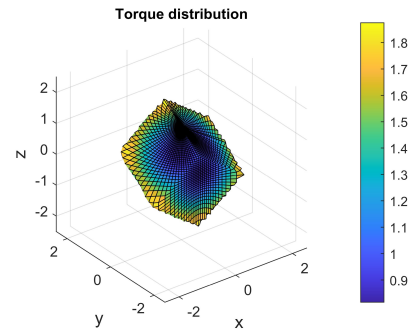


Fig. 4. Maximum torque distribution in the pyramid configuration with a skew angle of 35 degrees of the reaction wheels, assuming failure of the fourth reaction wheel.

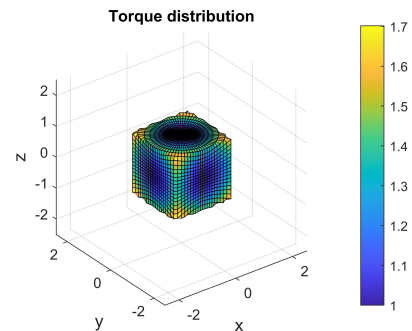


Fig. 5. Maximum torque distribution in three orthogonal reaction wheels.

4. MINIMUM-TIME EIGEN-AXIS REPOINTING MANEUVER

In a general attitude-change maneuver, the eigen-axis is uniquely constrained to a single fixed direction. In contrast, for a repointing maneuver, the rotation axis can be arbitrarily chosen within the plane of symmetry. Since the geometric distribution of achievable torques in both three- and four-reaction-wheel configurations is non-uniform (as depicted in Figs. 3–5), the eigen-axis vector can be optimized to minimize the required maneuver time. The geometric analysis demonstrates that, in a pyramidal reaction-wheel configuration, the time-optimal eigen-axis does not necessarily lie perpendicular to the plane formed by the initial and final attitude vectors. This result, derived within the framework of the underlying assumptions and the model presented in this research, underscores the direct influence of the non-uniform torque envelope on the selection of the optimal rotation axis.

The objective of investigating different reaction-wheel arrangements in this section is not to conduct a direct performance comparison, but rather to analyze how the geometry of the achievable torque set dictates the structure of the minimum-time problem, and to establish the conditions under which the repointing maneuver can be effectively reduced to a minimum-time control problem.

As stated in [15], the eigen-axis that enables the eigen-axis repointing maneuver lies in a plane with respect to which the pre- and post-maneuver pointing directions are symmetric. To uniquely determine this plane, it suffices to define its normal unit vector. Assuming unit lengths for the initial and final pointing vectors, denoted by $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively, the unit normal vector to the plane of symmetry is given by:

$$\mathbf{n} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \quad (23)$$

The spatial orientation of the unit normal vector $\mathbf{n} = [n_1, n_2, n_3]^T$ is parameterized by the spherical angles θ and Φ as:

$$\theta = \sin^{-1}(n_3), \quad \Phi = \begin{cases} \tan^{-1}\left(\frac{n_2}{n_1}\right) & n_1 \geq 0 \\ \tan^{-1}\left(\frac{n_2}{n_1}\right) + \pi & n_1 < 0 \end{cases} \quad (24)$$

The parametric equation of the unit circle defining all admissible eigen-axis directions $\mathbf{e}$ within the plane of symmetry of $\mathbf{r}_1$ and $\mathbf{r}_2$, parameterized by $\varepsilon \in [0, 2\pi)$, is

formulated as:

$$\mathbf{e}(\varepsilon) = (\mathbf{R}_x(\theta)\mathbf{R}_z(\Phi))^{-1} \begin{bmatrix} \cos \varepsilon \\ \sin \varepsilon \\ 0 \end{bmatrix} \quad (25)$$

where the transformation matrices $\mathbf{R}_x(\theta)$ and $\mathbf{R}_z(\Phi)$ are defined as:

$$\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & \cos \theta \\ 0 & -\cos \theta & \sin \theta \end{bmatrix} \quad (26)$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} -\sin \Phi & \cos \Phi & 0 \\ -\cos \Phi & -\sin \Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (27)$$

Figure 6 illustrates the geometric locus of the eigen-axis within the plane of symmetry that accomplishes the repointing maneuver from A to B. In this figure, ψ denotes the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$, and ε represents the angle between the eigen-axis and the plane passing through $\mathbf{r}_1$ and $\mathbf{r}_2$.

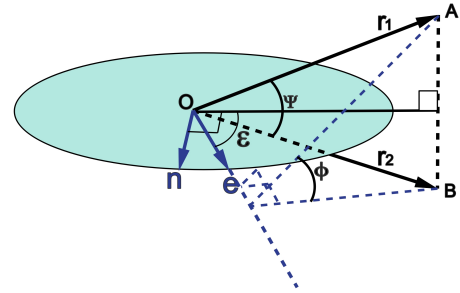


Fig. 6. The geometric locus of the Eigen-axis in the symmetry plane that can perform the repointing maneuver from A to B.

Since the plane OAB is perpendicular to the eigen-axis $\mathbf{e}$, the rotation angle associated with the eigen-axis repointing maneuver, denoted by ϕ , is determined by the following geometric relation:

$$\phi = 2 \tan^{-1} \left(\frac{\tan(\psi/2)}{\sin \varepsilon} \right) \quad (28)$$

According to Eq. (28), the minimum rotation angle $\phi = \psi$ in the eigen-axis repointing maneuver is achieved when $\varepsilon = 90^\circ$, corresponding to the case where the eigen-axis is perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$. Conversely, the maximum rotation angle $\phi = 180^\circ$ occurs when $\varepsilon = 0^\circ$, in which the eigen-axis lies entirely within the plane spanned by $\mathbf{r}_1$ and $\mathbf{r}_2$.

Numerical analysis indicates that the minimum-time eigen-axis repointing maneuver does not necessarily coincide with the minimum rotation angle ($\phi = \psi$). To substantiate this behavior, consider an illustrative case study with the initial and target pointing vectors defined as:

$$\mathbf{r}_1 = [0.784, 0.365, 0.500]^T, \quad \mathbf{r}_2 = [0, 0, 1]^T \quad (29)$$

The corresponding simulation results are summarized in Table 1.

Table 1. Optimization results in the three introduced configurations.

	Three RWs	Pyramid, 4th wheel failed	Four RWs
ϵ (deg)	53.99	75.89	75.89
$\mathbf{e}$	$\begin{bmatrix} 0.60 \\ -0.60 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.52 \\ -0.82 \\ 0.21 \end{bmatrix}$	$\begin{bmatrix} 0.52 \\ -0.82 \\ 0.21 \end{bmatrix}$
ϕ (deg)	71.02	61.52	61.52
$u_{e \max}$ (N·m)	1.642	0.989	1.979
t_f (s)	1.73	2.08	1.47

Figs. 7–9 show the curves of the rotation angle, maneuver time, and maximum torque as functions of the angle ϵ for the introduced configurations:

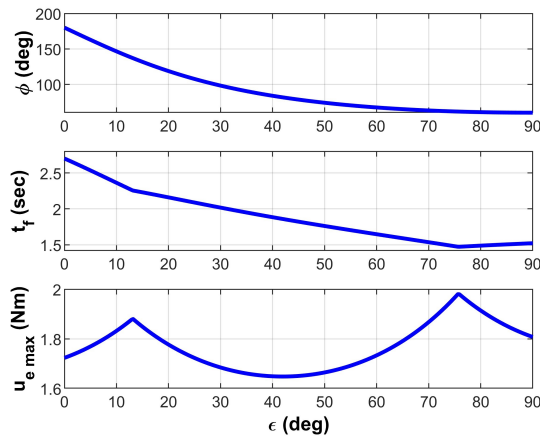


Fig. 7. Rotation angle, maneuver time, and maximum torque for changes in angle ϵ in the pyramidal configuration.

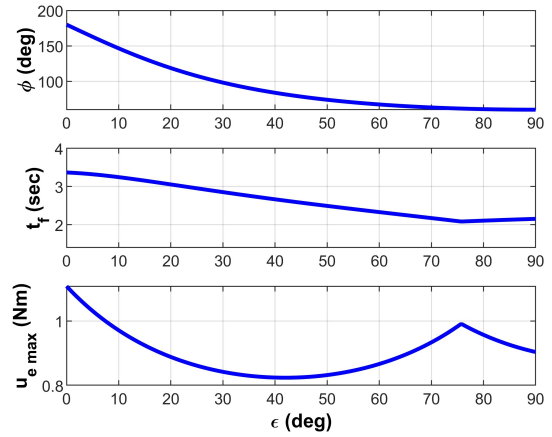


Fig. 8. Rotation angle, maneuver time, and maximum torque for changes in angle ϵ in the pyramidal configuration, assuming that the fourth wheel has failed.

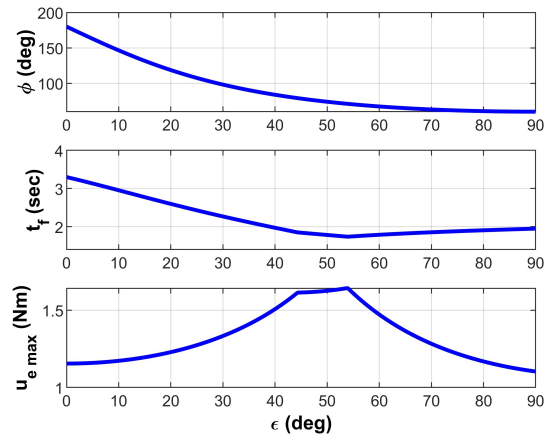


Fig. 9. Rotation angle, maneuver time, and maximum torque for changes in angle ϵ in the three-orthogonal-reaction-wheel configuration.

Figures 7–9 illustrate the variation of the maneuver time, t_f , as a function of the eigen-axis angle, ϵ . The minimum maneuver time is achieved at an angle below 90° , whereas for $\epsilon = 90^\circ$, the required time is longer than the optimum value. Given that minimizing the maneuver duration directly translates into enhanced spacecraft agility, the results demonstrate that an optimal selection of the eigen-axis orientation can reduce the maneuver time by approximately 8%. This reduction significantly enhances the maneuver agility, thereby improving the spacecraft's capability to execute rapid repointing and slewing maneuvers.

5. MINIMUM-TIME CONTROLLER FOR THE EIGEN-AXIS REPOINTING MANEUVER

For an axisymmetric or spherically symmetric spacecraft (i.e., $I_1 = I_2 = I_3 = I$), the governing rotational dynamic equation presented in Eq. (1) simplifies directly to:

$$\dot{\boldsymbol{\omega}} = \frac{\mathbf{u}}{I} = \mathbf{a} \quad (30)$$

where $\mathbf{a}$ denotes the angular acceleration vector. Following the formulation in [16], The coupled rotational kinematics and dynamics of a rigid spacecraft executing a fixed eigen-axis maneuver can be formulated as a second-order nonlinear system. By employing an appropriate state transformation, this system can be cast into an equivalent linear second-order form. Under these conditions, the minimum-time control law for the eigen-axis maneuver of a symmetric spacecraft is defined as follows:

$$u_i = -c_i \operatorname{sgn} \left(4 \tan^{-1}(\sigma_e) - 4 \tan^{-1}(\sigma_f) + \frac{I\omega_e |\omega_e|}{2u_{e\max}} \right), \quad i = 1, 2, 3, 4 \quad (31)$$

where σ_e and ω_e denote the scalar projections of the attitude parameter vector (Modified Rodrigues Parameters, MRPs), $\boldsymbol{\sigma}$, and the spacecraft angular velocity vector, $\boldsymbol{\omega}$, along the eigen-axis $\mathbf{e}$, respectively:

$$\sigma_e = \mathbf{e}^T \boldsymbol{\sigma}, \quad \omega_e = \mathbf{e}^T \boldsymbol{\omega} \quad (32)$$

Furthermore, σ_f represents the projected terminal attitude corresponding to the target orientation, expressed as:

$$\sigma_f = \mathbf{e}^T \mathbf{e} \tan \left(\frac{\phi}{4} \right) = \tan \left(\frac{\phi}{4} \right) \quad (33)$$

recalling that $\mathbf{e}^T \mathbf{e} = 1$. Here, ϕ denotes the total required principal rotation angle, and $u_{e\max}$ represents the maximum control torque deliverable along the designated eigen-axis $\mathbf{e}$.

From an implementation perspective, the bang–bang control law formulated herein is derived under the assumption of an ideal actuator model. In practical aerospace systems, however, nonideal physical phenomena, including actuator response latency, speed saturation limits, inevitably introduce deviations from the theoretical switching surface. Consequently, deploying this control law on actual flight hardware necessitates an intermediate actuator management and allocation layer to accommodate hardware limitations while preserving

the near optimal switching structure. Characterizing the influence of these practical constraints on maneuver performance constitutes a key objective for future extensions of this work.

6. NUMERICAL SIMULATION AND RESULTS

Based on the theoretical formulation developed in the preceding sections, the time-optimal control strategy is implemented and validated across three actuator configurations:

- A standard four-reaction-wheel pyramid cluster
- A degraded pyramid cluster experiencing a single-reaction-wheel failure
- A conventional three-axis orthogonal reaction wheel assembly

The simulation is conducted for a spherically symmetric spacecraft characterized by the inertia matrix and maximum deliverable wheel torque:

$$\mathbf{I} = \operatorname{diag}(1, 1, 1) \text{ kg} \cdot \text{m}^2, \quad \text{and} \quad u_m = 1 \text{ N} \cdot \text{m} \quad (34)$$

for each reaction wheel. The repointing maneuver is defined between the initial and target unit pointing vectors:

$$\mathbf{r}_1 = \begin{bmatrix} 0.784 \\ 0.365 \\ 0.500 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (35)$$

For each configuration, the optimal eigen-axis orientation and the corresponding principal slew angle are adopted from the optimization results obtained in the prior section. Subsequently, the individual wheel control coefficients (c_i) required to track the time-optimal eigen-axis trajectory are synthesized via the proposed allocation algorithm.

The resulting wheel control coefficients for each configuration are summarized in Table 2.

Table 2. Control coefficients used in the bang–bang controller for the three introduced configurations.

	Three orthogonal RWs	Pyramid, 4th wheel failed	Four RWs
c_1	0.608	0.994	0.989
c_2	−0.609	−1	−1
c_3	0.509	0.366	−0.267
c_4	—	—	1

Figures 10–18 present the dynamic response and performance metrics for the three considered actuator

configurations. Specifically, these profiles illustrate the time histories of the principal rotation angle, attitude error parameters, and the synthesized wheel control torques throughout the eigen-axis repointing maneuver.

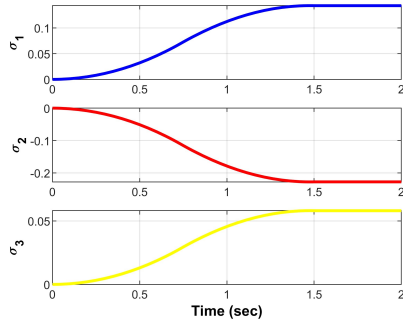


Fig. 10. Modified Rodrigues vector expressing the satellite attitude in the repointing maneuver in the pyramid configuration.

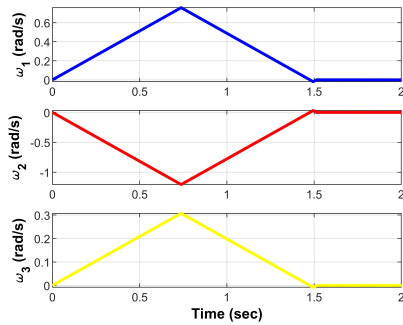


Fig. 11. Angular velocity vector of the satellite body in the repointing maneuver in the pyramid configuration.

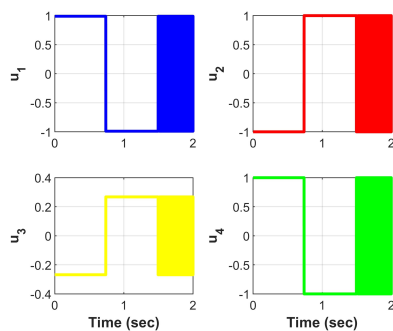


Fig. 12. Time history of the reaction wheel torques in the repointing maneuver in the pyramidal structure.

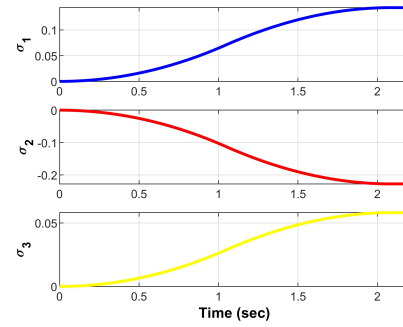


Fig. 13. Modified Rodrigues vector expressing the satellite attitude in the repointing maneuver in the pyramid configuration, assuming failure of the fourth reaction wheel.

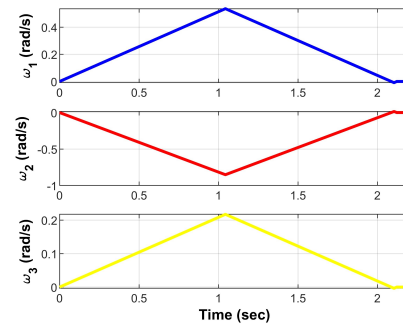


Fig. 14. Angular velocity vector of the satellite body in the repointing maneuver in the pyramid configuration, assuming failure of the fourth reaction wheel.

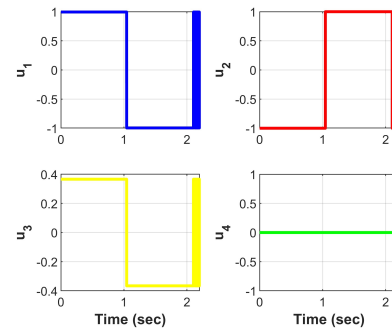


Fig. 15. Time history of the reaction wheel torques in the repointing maneuver in the pyramidal structure, assuming failure of the fourth reaction wheel.

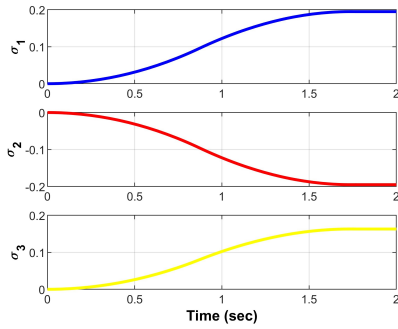


Fig. 16. Modified Rodrigues vector expressing the satellite attitude in the repointing maneuver in the three-orthogonal-reaction-wheel configuration.

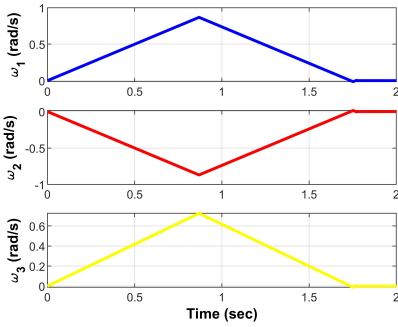


Fig. 17. Angular velocity vector of the satellite body in the repointing maneuver in the three-orthogonal-reaction-wheel configuration.

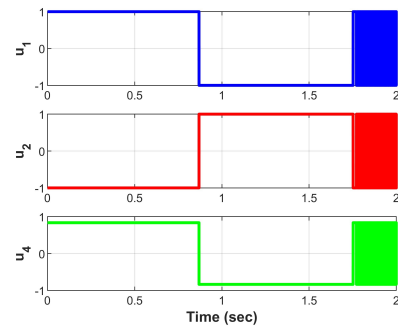


Fig. 18. Time history of the reaction wheel torques in the repointing maneuver in the three-orthogonal-reaction-wheel configuration.

7. CONCLUSION

In this paper, the problem of minimum-time eigen-axis repointing was investigated for symmetric spacecraft. The actuators considered included four reaction wheels in a pyramidal configuration, the pyramidal configuration with one wheel failed, and three orthogonal reaction wheels. Using a geometric approach, the maximum torque along any arbitrary direction and the control coefficient of each wheel for applying the minimum-time control were obtained. In addition, the geometric torque distribution was plotted for the three configurations, showing that, in the pyramidal configuration, this distribution takes the shape of a regular dodecagon. The results showed that the non-uniformity of the maximum torque means that, in the eigen-axis repointing maneuver, the eigen-axis perpendicular to the initial and final pointing vectors is not necessarily optimal, even though it corresponds to the minimum rotation angle. Therefore, the optimal eigen-axis and the maximum torque were computed and used in the minimum-time eigen-axis control law. In the numerical experiment, it was observed that the optimal eigen-axis lies along the edges of the regular dodecagon. Furthermore, the geometric torque distribution under the condition of a single wheel failure was examined, and the method for applying the minimum-time control law under these conditions was presented. These results show that the proposed method can improve spacecraft agility by about 8% in real missions.

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