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Model Predictive Control of Spacecraft Roto-Translational Relative Motion Using Dual-Quaternion Model

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ABSTRACT

Integrated modelling of spacecraft roto-translational relative motion is an emerging topic of interest for multi-agent space systems design, analysis and control. Dual-quaternion is a satisfactory solution to this complex problem. Model Predictive Control (MPC) is, on the other hand, a powerful model-based control methodology with known superiorities over other modern control approaches used in the field of spacecraft control. Embedding the dual-quaternion model in the MPC approach for sophisticated systems, thus yields a valuable control framework. There are, however, some difficulties involved in this proposition that need to be addressed. This matter is scrutinized in the present work. To this aim, the concept of dual-quaternion and relevant tools are first reviewed. Subsequently, potential interfaces between dual-quaternion expressions and MPC framework are highlighted. Accordingly, a piecewise affine MPC scheme based on the dual-quaternion model is introduced and further developed in several aspects for space roto-translational relative control missions. First, it has been improved for zero steady-state error. Furthermore, it has been developed to explicitly deal with reaction wheels as attitude actuators, in terms of limitations as well as related costs. The efficacy of the proposed integrated scheme is assessed by comprehensive simulations.

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NOMENCLATURE

$\hat{q}$	Dual-quaternion
$\boldsymbol{\varepsilon}$	Dual unit
$\hat{\otimes}$, $\widehat{\circledcirc}$	Dual products of the first type
$\hat{\times}$, $\hat{\circ}$	Dual products of the second type
$\hat{\omega}$	Dual angular velocity
$\hat{u}$	Dual input
$\hat{J}$	Dual inertia

1. INTRODUCTION

Space multi-vehicle systems have attracted research interests in recent years. Maximum reliability and scalability and minimum complexity and cost could be simultaneously achieved by multi spacecraft systems. To exploit these advantages, however, a multi-agent system should be controlled in a decentralized and coordinated manner [1]. More specifically, each spacecraft should be able to control its roto-translational states relative to its neighboring vehicles. Modelling of the spacecraft 6-DOF relative dynamics is thus the cornerstone.

Modelling the spacecraft relative motion has a considerable body of knowledge. Most of the methods are, however, dedicated to either relative orbit or relative attitude. To achieve accurate 6-DOF control performance, modelling the roto-translational relative motion in a unified framework is essential. Dual-quaternion [2], geometric mechanics or Lie algebra [3], dual Lie algebra [4] and Lagrangian method [5] are significant coupled roto-translational methods. Among these, the dual-quaternion method benefits from considerable advantages. Dual-quaternion is a minimal and singularity-free representation of the roto-translational space, i.e., $\mathbb{SE}(3)$, which includes the cross coupling between orbital and attitude dynamics and provides a unified expression for the 6-DOF dynamics. Furthermore, it is a direct generalization of the well-known quaternion concepts and tools for rotational dynamics.

The model predictive control (MPC) approach, on the other hand, is a popular and widely used modern control paradigm due to its exclusive advantages [6]. Providing an optimal state trajectory in a finite horizon and taking benefit of the knowledge of the future reference trajectory are among the superiorities of MPC over other control methods. More specifically, direct inclusion of heterogeneous constraints in the control law is a key advantage of MPC for most of the real-world applications, including the spacecraft control area which faces diverse and strict constraints on the states and control input. The remarkable progress in the field of microelectronics in recent decades which has led to the considerable enhancement of the available on-board computational capabilities within the limited electrical power requirements, has completely provided the groundwork to realize MPC for spacecraft control.

MPC is a model-based control framework in the sense that a mathematical model of the plant explicitly appears in the controller body. The Complementary combination of the dual-quaternion modelling paradigm and the MPC control approach could thus provide a comprehensive framework for multi-spacecraft control. Investigating the literature shows, however, that only few researches have yet been conducted in which the dual-quaternion model of spacecraft relative 6-DOF dynamics is applied in the MPC framework. The reason may rely on the challenges of matching the dual-quaternion model nature with the MPC inherent requirements. Considering the mathematical essence of the problem, the difficulties could be identified and summarized as follows: how to define a scalar cost function based on dual states; how to discretize the dual dynamic equations; and how to develop a closed-form predictive model based on dual operators.

The outstanding work of Mesbahi et al [7] is, to the best of the author's knowledge, the only published research aiming to integrate MPC and dual-quaternion model of spacecraft relative motion, which introduces a piecewise affine structure. However, the mentioned work possesses some shortcomings. The proposed MPC rule contains explice input term and thus, is biased (leads to a constant steady-state error). Furthermore, and more important, the control law lacks sufficient rigor from the actuation perspective. More specifically, it is based on hypothetical ideal control inputs and does not take into account the practical considerations for the actuators. Uncertainties and disturbances are also neglected.

This paper aims to take a step forward. To this end, the fundamental concepts and tools of the dual-quaternion paradigm are first reviewed in a way that reveals their potential connections with MPC. Thereupon, the piecewise affine MPC scheme based on the dual-quaternion model is discussed. These steps have paved the way for presenting the main contributions of the paper, which are summarized as follows:

1) The dual-quaternion-based MPC rule has been improved for zero steady-state error, i.e., to be unbiased (section 3.2).
2) The control framework has been developed to explicitly deal with reaction wheels as attitude actuators. Specifically, the dual-quaternion-based MPC has been augmented to include reaction wheels angular momentum limitations as constraints (section 3.3).
3) Furthermore, the dual-quaternion-based MPC has been enhanced to efficiently expend the capacity of the reaction wheels by including a relevant cost term in the performance index (section 3.4).
4) The developments have been verified via comprehensive simulation studies (section 4).

The rest of the paper is organized as follows: The essentials of the spacecraft relative roto-translational modelling based on the dual-quaternion technique are reviewed in section 2. The developed dual-quaternion based MPC is provided in section 3. In Section 4 the proposed framework is verified by simulation. Finally, concluding remarks and future research directions are provided in Section 5.

2. FUNDAMENTALS OF DUAL-QUATERNION APPROACH FOR SPACECRAFT 6-DOF RELATIVE DYNAMICS MODELLING

Introducing the dual-quaternion mathematics and its consequent dynamical developments in depth is beyond the scope of this paper. The main results are thus summarized here, to be used in the rest of the paper. Further details can be found in [8,9].

2.1 Concepts and Tools

The notion of "dual-quaternion" is an elegant extension for the well-known quaternion concept. As the unit quaternion is a minimal representation of the rotation group, $\mathbb{SO}(3)$, the unit dual-quaternion parameterizes the translation-rotation group, $\mathbb{SE}(3)$, in a minimal and integrated framework [10]. In other words, the dual-quaternion framework has the extra capacity to embed and express translational information along with rotational data.

Introducing the dual-quaternion mathematics and its consequent dynamical developments in depth is beyond the scope of this paper. The main results are thus summarized here, to be used in the rest of the paper, and the details are referred to [8,9].

The definition of dual-quaternions can be best established from one step back, i.e., dual-numbers. A dual-number, in its most general sense, is defined as a special combination of two general vectors of the same length. Specifically:

$$\hat{\boldsymbol{a}} \triangleq \boldsymbol{a}_r + \varepsilon \boldsymbol{a}_d \in \widehat{\mathbb{R}}^n \quad , \quad \boldsymbol{a}_r, \boldsymbol{a}_d \in \mathbb{R}^n \tag{1}$$

$\boldsymbol{a}_r$ and $\boldsymbol{a}_d$ are called the real part and dual part, respectively, and ε is the dual unit defined by the following unconventional properties:

$$\begin{aligned} &\varepsilon \neq 0\,, \varepsilon^2 = 0 \\ &\Longrightarrow \forall k \in \{2, 3, \dots\}: \varepsilon^k = 0 \end{aligned} \tag{2}$$

For $n = 1, 2$ and 3 the dual-number is called dual-scalar, dual-vector and dual-quaternion, respectively. In the case of dual-quaternion, i.e., $\hat{\boldsymbol{q}} \in \widehat{\mathbb{Q}} = \widehat{\mathbb{R}}^4$, there are additional representations, due to the notion of quaternion:

$$\begin{aligned} \hat{\boldsymbol{q}} &\triangleq \boldsymbol{q}_r + \varepsilon \boldsymbol{q}_d \\ &= [q_{r0} \quad \boldsymbol{q}_{rv}^T]^T + \varepsilon [q_{d0} \quad \boldsymbol{q}_{dv}^T]^T \\ &= [\hat{q}_0 \quad \hat{\boldsymbol{q}}_v^T]^T \\ &= [q_{r0} + \varepsilon q_{d0} \quad \boldsymbol{q}_{rv}^T + \varepsilon \boldsymbol{q}_{dv}^T]^T \\ &= [\hat{q}_0 \quad \hat{q}_1 \quad \hat{q}_2 \quad \hat{q}_3]^T \end{aligned} \tag{3}$$

where $\boldsymbol{q}_r$ and $\boldsymbol{q}_d$ are regular quaternions. q_{r0} and $\boldsymbol{q}_{rv}$ are respectively the scalar part and the vector part of $\boldsymbol{q}_r$. Of course, the same expression holds for q_{d0} and $\boldsymbol{q}_{dv}$ with respect to $\boldsymbol{q}_d$. Furthermore, $\hat{q}_0 \in \widehat{\mathbb{R}}$ and $\hat{\boldsymbol{q}}_v \in \widehat{\mathbb{R}}^3 = \widehat{\mathbb{Q}}_v$ are dual-scalar part and dual-vector part, respectively.

As long as the operations on dual-numbers are linear, they can alternatively be represented as a regular vector in $\mathbb{R}^{2n}$ Euclidean space. In particular, a dual-quaternion can be considered as follows [7]:

$$\hat{\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{q}_r \\ \boldsymbol{q}_d \end{bmatrix}_{8\times1} \tag{4}$$

This expression, which is free from the dual unit ε, is quite useful in formulating an MPC rule based on a dual-quaternion based model.

To express the roto-translational kinematics and dynamics of spacecraft by dual-quaternions, two pairs of product operators are defined. These operators are direct extensions of regular quaternion operators and thus, are similar in form to their quaternion counterparts. It is noteworthy that there is no universally accepted notation for the product operators in the literature, even for regular quaternions. Here, adapting various alternatives, a comprehensive and consistent notation is provided.

The product of the first type is defined as a pair of operators, as follows:

$$\hat{\boldsymbol{q}} \widehat{\otimes} \hat{\boldsymbol{p}} \triangleq \begin{bmatrix} \hat{q}_0\hat{p}_0 - \hat{\boldsymbol{q}}_v^T\hat{\boldsymbol{p}}_v \\ \hat{q}_0\hat{\boldsymbol{p}}_v + \hat{p}_0\hat{\boldsymbol{q}}_v + \hat{\boldsymbol{q}}_v \times \hat{\boldsymbol{p}}_v \end{bmatrix} \tag{5}$$

$$\hat{\boldsymbol{q}} \widehat{\circledcirc} \hat{\boldsymbol{p}} \triangleq \begin{bmatrix} \hat{q}_0\hat{p}_0 - \hat{\boldsymbol{q}}_v^T\hat{\boldsymbol{p}}_v \\ \hat{q}_0\hat{\boldsymbol{p}}_v + \hat{p}_0\hat{\boldsymbol{q}}_v - \hat{\boldsymbol{q}}_v \times \hat{\boldsymbol{p}}_v \end{bmatrix} \tag{6}$$

where, $\hat{\boldsymbol{q}}$ and $\hat{\boldsymbol{p}}$ are arbitrary dual-quaternions for which the structure and elements are stated in (3). These expressions can be recast in terms of dual-matrix multiplications as:

$$\hat{\boldsymbol{q}} \widehat{\otimes} \hat{\boldsymbol{p}} = [\hat{\boldsymbol{q}}]_{\widehat{\otimes}} \hat{\boldsymbol{p}} \tag{7}$$

$$\hat{\boldsymbol{q}} \widehat{\circledcirc} \hat{\boldsymbol{p}} = [\hat{\boldsymbol{q}}]_{\widehat{\circledcirc}} \hat{\boldsymbol{p}} \tag{8}$$

where, the dual-matrix coefficients are given as follows:

$$[\hat{\boldsymbol{q}}]_{\widehat{\otimes}} \triangleq \begin{bmatrix} \hat{q}_0 & -\hat{\boldsymbol{q}}_v^T \\ \hat{\boldsymbol{q}}_v & \hat{\boldsymbol{q}}_v^\times + \hat{q}_0\boldsymbol{I}_3 \end{bmatrix}_{8\times8} \tag{9}$$

$$[\hat{\boldsymbol{q}}]_{\widehat{\circledcirc}} \triangleq \begin{bmatrix} \hat{q}_0 & -\hat{\boldsymbol{q}}_v^T \\ \hat{\boldsymbol{q}}_v & -\hat{\boldsymbol{q}}_v^\times + \hat{q}_0\boldsymbol{I}_3 \end{bmatrix} \tag{10}$$

Here, $\hat{\boldsymbol{q}}_v^\times$ is the skew-symmetric dual-matrix corresponding to the dual-vector $\hat{\boldsymbol{q}}_v$, given as follows:

$$\hat{\boldsymbol{q}}_v^\times = \begin{bmatrix} 0 & -\hat{q}_3 & \hat{q}_2 \\ \hat{q}_3 & 0 & -\hat{q}_1 \\ -\hat{q}_2 & \hat{q}_1 & 0 \end{bmatrix} \tag{11}$$

The relation between the two operators is:

$$\hat{\boldsymbol{q}} \widehat{\otimes} \hat{\boldsymbol{p}} = \hat{\boldsymbol{p}} \widehat{\circledcirc} \hat{\boldsymbol{q}} \tag{12}$$

Observe that in (9) and (10), the matrices are not regular, because they contain the dual unit, ε. As will be seen in the next section, this issue poses significant challenges for an MPC rule to be formulated. Nevertheless, since the operations are linear, alternative expressions could be provided to eliminate the drawback. Specifically, if dual-quaternions are expressed as regular vectors of the form (4), the following expressions are identical to (5) and (6), respectively:

$$\hat{\boldsymbol{q}} \widehat{\otimes} \hat{\boldsymbol{p}} = \begin{bmatrix} [\boldsymbol{q}_r]_\otimes & \boldsymbol{0}_{4\times4} \\ [\boldsymbol{q}_d]_\otimes & [\boldsymbol{q}_r]_\otimes \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_r \\ \boldsymbol{p}_d \end{bmatrix} \tag{13}$$

$$\hat{\boldsymbol{q}} \widehat{\circledcirc} \hat{\boldsymbol{p}} = \begin{bmatrix} [\boldsymbol{q}_r]_\odot & \boldsymbol{0}_{4\times4} \\ [\boldsymbol{q}_d]_\odot & [\boldsymbol{q}_r]_\odot \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_r \\ \boldsymbol{p}_d \end{bmatrix} \tag{14}$$

where $[\boldsymbol{q}]_\otimes$ and $[\boldsymbol{q}]_\odot$ are regular quaternion counterparts of $[\hat{\boldsymbol{q}}]_{\widehat{\otimes}}$ and $[\hat{\boldsymbol{q}}]_{\widehat{\circledcirc}}$ as formulated in (9) and (10), respectively:

$$[\boldsymbol{q}]_\otimes \triangleq \begin{bmatrix} q_0 & -\boldsymbol{q}_v^T \\ \boldsymbol{q}_v & \boldsymbol{q}_v^\times + q_0\boldsymbol{I}_3 \end{bmatrix} \tag{15}$$

$$[\boldsymbol{q}]_\odot \triangleq \begin{bmatrix} q_0 & -\boldsymbol{q}_v^T \\ \boldsymbol{q}_v & -\boldsymbol{q}_v^\times + q_0\boldsymbol{I}_3 \end{bmatrix} \tag{16}$$

where:

$$\boldsymbol{q}_v^\times \triangleq \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{17}$$

Observe that in (13) and (14), $[\hat{\boldsymbol{q}}]_{\widehat{\otimes}}$ and $[\hat{\boldsymbol{q}}]_{\widehat{\circledcirc}}$ are regular matrices.

The definition of $\widehat{\otimes}$ ($\widehat{\circledcirc}$) becomes more comprehensive by extending it to include dual-vectors as well. The dual-quaternion representation of a dual-vector, $\hat{\boldsymbol{a}} = \boldsymbol{a}_r + \varepsilon\boldsymbol{a}_d$, is defined as follows:

$$\begin{aligned} \hat{\boldsymbol{a}}^q &\triangleq \boldsymbol{a}_r^q + \varepsilon\boldsymbol{a}_d^q \\ &\triangleq [0 \quad \boldsymbol{a}_r^T]^T + \varepsilon[0 \quad \boldsymbol{a}_d^T]^T \end{aligned} \tag{18}$$

Accordingly, for a dual-quaternion and a dual-vector, the mentioned convention could be extended as follows:

$$\hat{\boldsymbol{q}} \widehat{\otimes} \hat{\boldsymbol{a}} \triangleq \hat{\boldsymbol{q}} \widehat{\otimes} \hat{\boldsymbol{a}}^q \tag{19}$$

For dynamics formulation, we need another type of product operator as well. The product pair of the second type is defined as follows:

$$\hat{\boldsymbol{q}} \,\hat{\times}\, \hat{\boldsymbol{p}} \triangleq \begin{bmatrix} \hat{0} \\ \hat{q}_0\hat{\boldsymbol{p}}_v + \hat{p}_0\hat{\boldsymbol{q}}_v + \hat{\boldsymbol{q}}_v \times \hat{\boldsymbol{p}}_v \end{bmatrix} \tag{20}$$

$$\hat{\boldsymbol{q}} \,\hat{\times}\, \hat{\boldsymbol{p}} = [\hat{\boldsymbol{q}}]_{\hat{\times}}\hat{\boldsymbol{p}} \tag{21}$$

$$[\hat{\boldsymbol{q}}]_{\hat{\times}} \triangleq \begin{bmatrix} \hat{0} & \hat{\boldsymbol{0}} \\ \hat{\boldsymbol{q}}_v & \hat{\boldsymbol{q}}_v^{\times} + \hat{q}_0\boldsymbol{I}_3 \end{bmatrix} \tag{22}$$

$$\hat{\boldsymbol{q}} \,\hat{\circ}\, \hat{\boldsymbol{p}} \triangleq \begin{bmatrix} \hat{0} \\ \hat{q}_0\hat{\boldsymbol{p}}_v + \hat{p}_0\hat{\boldsymbol{q}}_v - \hat{\boldsymbol{q}}_v \times \hat{\boldsymbol{p}}_v \end{bmatrix} \tag{23}$$

$$\hat{\boldsymbol{q}} \,\hat{\circ}\, \hat{\boldsymbol{p}} \triangleq [\hat{\boldsymbol{q}}]_{\hat{\circ}}\hat{\boldsymbol{p}} \tag{24}$$

$$[\hat{\boldsymbol{q}}]_{\hat{\circ}} \triangleq \begin{bmatrix} \hat{0} & \hat{\boldsymbol{0}} \\ \hat{\boldsymbol{q}}_v & -\hat{\boldsymbol{q}}_v^{\times} + \hat{q}_0\boldsymbol{I}_3 \end{bmatrix} \tag{25}$$

$$\hat{\boldsymbol{q}} \,\hat{\times}\, \hat{\boldsymbol{p}} = \hat{\boldsymbol{p}} \,\hat{\circ}\, \hat{\boldsymbol{q}} \tag{26}$$

In the same manner, the product of the second type for two dual-vectors is defined as the product of their dual-quaternion counterparts:

$$\hat{\boldsymbol{a}} \,\hat{\times}\, \hat{\boldsymbol{b}} \triangleq \hat{\boldsymbol{a}}^q \,\hat{\times}\, \hat{\boldsymbol{b}}^q \tag{27}$$

In this equation, $\hat{\boldsymbol{a}}^q$ and $\hat{\boldsymbol{b}}^q$ are the dual-quaternion equivalents for the dual-vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$, respectively (refer to (18)).

The linearity of the operator, again leads to a more favorable formulation, free from the dual factor ε, as follows:

$$\begin{aligned} \hat{\boldsymbol{q}} \,\hat{\times}\, \hat{\boldsymbol{p}} &= \begin{bmatrix} [\boldsymbol{q}_r]_{\times} & \boldsymbol{0}_{4\times4} \\ [\boldsymbol{q}_d]_{\times} & [\boldsymbol{q}_r]_{\times} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_r \\ \boldsymbol{p}_d \end{bmatrix} \\ &= [\hat{\boldsymbol{q}}]_{\hat{\times}}\hat{\boldsymbol{p}} \end{aligned} \tag{28}$$

$$\begin{aligned} \hat{\boldsymbol{q}} \,\hat{\circ}\, \hat{\boldsymbol{p}} &= \begin{bmatrix} [\boldsymbol{q}_r]_{\circ} & \boldsymbol{0}_{4\times4} \\ [\boldsymbol{q}_d]_{\circ} & [\boldsymbol{q}_r]_{\circ} \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_r \\ \boldsymbol{p}_d \end{bmatrix} \\ &= [\hat{\boldsymbol{q}}]_{\hat{\circ}}\hat{\boldsymbol{p}} \end{aligned} \tag{29}$$

Again, $[\hat{\boldsymbol{q}}]_{\hat{\times}}$ and $[\hat{\boldsymbol{q}}]_{\hat{\circ}}$ are defined as regular versions of their dual counterparts, as follows:

$$[\boldsymbol{q}]_{\times} \triangleq \begin{bmatrix} 0 & \boldsymbol{0}_{1\times3} \\ \boldsymbol{q}_v & \boldsymbol{q}_v^{\times} + q_0\boldsymbol{I}_3 \end{bmatrix} \tag{30}$$

$$\begin{aligned} [\boldsymbol{q}]_{\circ} &\triangleq \begin{bmatrix} 0 & \boldsymbol{0}_{1\times3} \\ \boldsymbol{q}_v & -\boldsymbol{q}_v^{\times} + q_0\boldsymbol{I}_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \boldsymbol{0}_{1\times3} \\ \boldsymbol{q}_v & \boldsymbol{q}_v^{\times T} + q_0\boldsymbol{I}_3 \end{bmatrix} \end{aligned} \tag{31}$$

2.2 Dual-Quaternion Model of Spacecraft Relative Roto-Translational Motion

Before going through the dynamics formulation, it should be mentioned that in our notation subscripts/superscripts $\boldsymbol{b}$ and $\boldsymbol{t}$ indicate the body frames/coordinate systems of the spacecraft of interest and the target spacecraft, respectively, subscript $\boldsymbol{bt}$ means "$\boldsymbol{b}$ with respect to $\boldsymbol{t}$" and a superscript states the coordinate system in which the variable is expressed.

The dual-quaternion notions and tools are built to provide a direct and precise extension of the quaternion notions and tools into the rotational-translational space. Upon appropriately established definitions for dual states, it is shown based on the notions and tools introduced in the previous section that the 6-DOF relative kinematic-kinetic equations are given as follows:

$$\dot{\hat{\boldsymbol{q}}}_{bt} = \frac{1}{2}\hat{\boldsymbol{q}}_{bt} \hat{\otimes} \hat{\boldsymbol{\omega}}_{bt}^b \tag{32}$$

$$\hat{\boldsymbol{J}}^b \dot{\hat{\boldsymbol{\omega}}}_{bt}^b = -\hat{\boldsymbol{\omega}}_{bt}^{\boldsymbol{b}} \,\hat{\times}\, \left(\hat{\boldsymbol{J}}^b \hat{\boldsymbol{\omega}}_{bt}^{\boldsymbol{b}}\right) + \hat{\boldsymbol{u}}^b \tag{33}$$

For these equations to be valid, the dual relative states are defined as follows:

$$\hat{\boldsymbol{\omega}}_{bt}^b = \boldsymbol{\omega}_{bt}^b + \varepsilon \boldsymbol{v}_{bt}^b \tag{34}$$

$$\begin{aligned} \hat{\boldsymbol{q}}_{bt} &= \boldsymbol{q}_{bt} + \varepsilon\frac{1}{2}\boldsymbol{q}_{bt} \otimes \boldsymbol{r}_{bt}^b \\ &= \boldsymbol{q}_{bt} + \varepsilon\frac{1}{2}\boldsymbol{r}_{bt}^t \otimes \boldsymbol{q}_{bt} \end{aligned} \tag{35}$$

where $\hat{\boldsymbol{\omega}}_{bt}^b$ and $\hat{\boldsymbol{q}}_{bt}$ are called the dual relative angular velocity vector and the dual relative quaternion, respectively. Note that $\hat{\boldsymbol{\omega}}_{bt}^b$ is a dual-vector in its nature. $\boldsymbol{\omega}_{bt}^b$ and $\boldsymbol{q}_{bt}$ are the rotational states, i.e., the regular relative angular velocity vector and regular relative quaternion, respectively. It could be observed that in the definition of the dual states, the translational states $\boldsymbol{r}_{bt}^b$ and $\boldsymbol{v}_{bt}^b$ which are the relative position vector and the relative linear velocity vector, respectively, are also involved along with their rotational counterparts.

Furthermore, $\hat{\boldsymbol{u}}^b$ in (33) is the dual input which is defined as follows:

$$\hat{\boldsymbol{u}}^b = \boldsymbol{f}^b + \varepsilon \boldsymbol{\tau}^b \tag{36}$$

where $\boldsymbol{\tau}^b$ and $\boldsymbol{f}^b$ are the net torque and net force, respectively, expressed in the body coordinates. Thus, a combination of rotational and translational excitations is present at the input side, as well. Again, observe that $\hat{\boldsymbol{u}}^b$ is a dual-vector.

Moreover, $\hat{\boldsymbol{J}}^b$ is the dual inertia given as follows:

$$\hat{\boldsymbol{J}}_b = m_b \boldsymbol{I}_3 \frac{d}{d\varepsilon} + \varepsilon \boldsymbol{J}_b = \begin{bmatrix} m_b \frac{d}{d\varepsilon} + \varepsilon J_{xx} & \varepsilon J_{xy} & \varepsilon J_{xz} \\ \varepsilon J_{xy} & m_b \frac{d}{d\varepsilon} + \varepsilon J_{yy} & \varepsilon J_{yz} \\ \varepsilon J_{xz} & \varepsilon J_{yz} & m_b \frac{d}{d\varepsilon} + \varepsilon J_{zz} \end{bmatrix} \quad (37)$$

in which $\boldsymbol{J}_b$ is the inertia matrix expressed in the body coordinates, and m_b is the mass. Again, the rotational and translational inertia quantities appear together. It should be noted that, based on the above definition, $\hat{\boldsymbol{J}}_b$ is indeed a dual operator.

Note: in the rest of the paper, explicit representation of the superscripts and subscripts for the state, input and inertia variables is waived, for simplicity.

As mentioned earlier, the dual unit ε and its consequences are undesirable phenomena in formulating MPC rules. In order to avoid this factor and, at the same time, benefit from the dual-quaternion framework, $\hat{\boldsymbol{q}}_{bt}$ should first be recast in the form of an eight elements vector, according to (4). The resultant expansion of (35) is given as follows:

$$\hat{\boldsymbol{q}} = \begin{bmatrix} q_0 & \boldsymbol{q}_v & \frac{-\boldsymbol{q}_v^T \boldsymbol{r}}{2} & \frac{q_0 \boldsymbol{r} + [\boldsymbol{q}_v]_\times \boldsymbol{r}}{2} \end{bmatrix}^T_{1\times8} \quad (38)$$

In the next step, the dual angular velocity and dual input which are both inherently dual-vectors, should be first converted to their corresponding dual-quaternion forms, according to (18), and then be expressed in the eight elements vector form, again based on (4). The eventual forms are as follow:

$$\begin{aligned} \hat{\boldsymbol{\omega}} &= \boldsymbol{\omega}^q + \varepsilon \boldsymbol{v}^q \\ &= [0 \quad \boldsymbol{\omega}^T]^T + \varepsilon [0 \quad \boldsymbol{v}^T]^T \\ &= [0 \quad \boldsymbol{\omega} \quad 0 \quad \boldsymbol{v}]^T_{1\times8} \end{aligned} \quad (39)$$

$$\begin{aligned} \hat{\boldsymbol{u}} &= \boldsymbol{f}^q + \boldsymbol{\varepsilon \tau}^q \\ &= [\boldsymbol{0} \quad \boldsymbol{f}^T]^T + \boldsymbol{\varepsilon} [\boldsymbol{0} \quad \boldsymbol{\tau}^T]^T \\ &= [0 \quad \boldsymbol{f} \quad 0 \quad \boldsymbol{\tau}]^T_{1\times8} \end{aligned} \quad (40)$$

In accordance with this formulation, the dual inertia is given as a regular matrix, as follows:

$$\hat{\boldsymbol{J}} = \begin{bmatrix} 0 & \boldsymbol{0}_{1\times3} & 1 & \boldsymbol{0}_{1\times3} \\ \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times1} & m\boldsymbol{I}_3 \\ 1 & \boldsymbol{0}_{1\times3} & 0 & \boldsymbol{0}_{1\times3} \\ \boldsymbol{0}_{3\times1} & \boldsymbol{J} & \boldsymbol{0}_{3\times1} & \boldsymbol{0}_{3\times3} \end{bmatrix}_{8\times8} \quad (41)$$

3. DUAL-QUATERNION-BASED PIECEWISE AFFINE MPC WITH REACTION WHEEL CONSIDERATIONS

3.1 Preliminary Formulation

Discretizing and recasting the 6-DOF relative dynamics equations (32)-(33) leads to the following "piecewise affine" model [7]:

$$\boldsymbol{x}(t+1) = \boldsymbol{A}_t \boldsymbol{x}(t) + \boldsymbol{B}_t \boldsymbol{u}(t) \quad (42)$$

where t is the discrete time and the extended roto-translational state is defined as a vector of two dual-quaternions as follows:

$$\boldsymbol{x}(t) \triangleq [\hat{\boldsymbol{\omega}}(t)^T \quad \hat{\boldsymbol{q}}(t)^T]^T_{16\times1} \quad (43)$$

Furthermore:

$$\boldsymbol{A}_t = \boldsymbol{M}_t^{-1} \boldsymbol{A}'_t \quad , \quad \boldsymbol{B}_t = \boldsymbol{M}_t^{-1} \boldsymbol{B}'_t \quad (44)$$

$$\boldsymbol{M}_t \triangleq \begin{bmatrix} \hat{\boldsymbol{J}} & \boldsymbol{0}_{8\times8} \\ -\frac{\Delta t}{2} [\hat{\boldsymbol{q}}(t)]_{\hat{\otimes}} & \boldsymbol{I}_8 \end{bmatrix}_{16\times16} \quad (45)$$

$$\boldsymbol{A}'_t \triangleq \begin{bmatrix} \hat{\boldsymbol{J}} - \Delta t [\hat{\boldsymbol{J}} \hat{\boldsymbol{\omega}}(t)]_{\hat{\circ}} & \boldsymbol{0}_{8\times8} \\ \boldsymbol{0}_{8\times8} & \boldsymbol{I}_8 \end{bmatrix}_{16\times16} \quad (46)$$

$$\boldsymbol{B}'_t \triangleq \begin{bmatrix} \Delta t \boldsymbol{I}_8 \\ \boldsymbol{0}_{8\times8} \end{bmatrix}_{16\times8} \quad (47)$$

Observe that (42) is a regular matrix equation, free from the dual unit ε. On the other hand, the form of $\boldsymbol{A}_t$ and $\boldsymbol{B}_t$ prevents the equation to be linear. However, it could be considered as an affine equation, in the sense that it is the sum of a factor of the state and a factor of the input. Since the equation is basically obtained from discretizing, it is more precise to say that (42) is the piecewise affine form of the 6-DOF relative dynamics equations.

According to the conventional structure of MPC, the predicted state vector and the future control vector for the current discrete instant t and the prediction-control horizon N are defined as follows:

$$\boldsymbol{X}_{16N\times1} = [\boldsymbol{x}^T(t+1) \quad \boldsymbol{x}^T(t+2) \quad \dots \quad \boldsymbol{x}^T(t+N)]^T \quad (48)$$

$$\boldsymbol{U}_{8N\times1} = [\boldsymbol{u}^T(t) \quad \boldsymbol{u}^T(t+1) \quad \dots \quad \boldsymbol{u}^T(t+N-1)]^T \quad (49)$$

By successive application of (42), the predictive model is obtained as follows:

$$\boldsymbol{X} = \boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \boldsymbol{\mathcal{B}}\boldsymbol{U} \tag{50}$$

in which:

$$\boldsymbol{\mathcal{A}}_{16N\times 16} = \begin{bmatrix} \boldsymbol{A}_t \\ \boldsymbol{A}_t^2 \\ \vdots \\ \boldsymbol{A}_t^N \end{bmatrix} \tag{51}$$

$$\boldsymbol{\mathcal{B}}_{16N\times 8N} = \begin{bmatrix} \boldsymbol{B}_t & \boldsymbol{0}_{16\times 8} & \boldsymbol{0}_{16\times 8} & \cdots & \boldsymbol{0}_{16\times 8} \\ \boldsymbol{A}_t\boldsymbol{B}_t & \boldsymbol{B}_t & \boldsymbol{0}_{16\times 8} & \cdots & \boldsymbol{0}_{16\times 8} \\ \vdots & & & & \\ \boldsymbol{A}_t^{N-1}\boldsymbol{B}_t & \boldsymbol{A}_t^{N-2}\boldsymbol{B}_t & \dots & \boldsymbol{A}_t\boldsymbol{B}_t & \boldsymbol{B}_t \end{bmatrix} \tag{52}$$

For the above equations, $\boldsymbol{A}_t$ and $\boldsymbol{B}_t$ are given in (44). Actually, (50) provides the future states based on the current measured state and the future inputs.

Following the prevalent approach in MPC, the quadratic cost function is defined as follows:

$$J = (\boldsymbol{X} - \boldsymbol{X_d})^T \boldsymbol{Q}(\boldsymbol{X} - \boldsymbol{X_d}) + \boldsymbol{U}^T\boldsymbol{R}\boldsymbol{U} \tag{53}$$

in which, $\boldsymbol{Q}_{16N\times 16N}$ and $\boldsymbol{R}_{8N\times 8N}$ are weight matrices. Furthermore, $\boldsymbol{X_d}$ is the desired state vector for the future time instants. The block elements of $\boldsymbol{X_d}$ are commonly considered identical to the current desired state.

The future input vector, $\boldsymbol{U}$, is obtained by minimizing the cost function. In order for the "quadratic programming" algorithms to be applied, the cost function should be rewritten as an explicit function of the optimization variable, $\boldsymbol{U}$, as follows:

$$J = \frac{1}{2}\boldsymbol{U}^T\boldsymbol{H}\boldsymbol{U} + \boldsymbol{f}^T\boldsymbol{U} + \boldsymbol{C} \tag{54}$$

where:

$$\boldsymbol{H} = 2(\boldsymbol{\mathcal{B}}^T\boldsymbol{Q}\boldsymbol{\mathcal{B}} + \boldsymbol{R}) \tag{55}$$

$$\boldsymbol{f} = 2\big(\boldsymbol{\mathcal{B}}^T\boldsymbol{Q}(\boldsymbol{\mathcal{A}}\boldsymbol{x}(t) - \boldsymbol{X_d})\big) \tag{56}$$

and the term $\boldsymbol{C}$ is the sum of all terms independent of $\boldsymbol{U}$ and ineffectual in the optimization.

Accordingly, the piecewise affine MPC rule based on the dual-quaternion model is given as follows:

$$\min_{\boldsymbol{U}} \quad J \quad \text{subject to constraints} \tag{57}$$

Observe that input and state constraints of various types can directly and explicitly be included in the control rule. Specifically, the input bound constraint is given as follows:

$$-\boldsymbol{U}_b \le \boldsymbol{U} \le \boldsymbol{U}_b \tag{58}$$

Upon solving the optimization problem at each time step, the future input vector, $\boldsymbol{U}$, is computed. According to the conventional structure of MPC, the first block element of this vector, $\boldsymbol{u}(t)$, which is the control input for the current time instant, is exerted to the plant, and the procedure is repeated in consecutive time steps.

3.2 Unbiased MPC

The first essential development for the designed piecewise affine MPC is to improve the predictive model based on the future input increments. It has been demonstrated that in the MPC structure, the input term in the cost function results in bias in the response. In order to resolve the problem, the cost function should be modified as follows:

$$J = (\boldsymbol{X} - \boldsymbol{X_d})^T \boldsymbol{Q}(\boldsymbol{X} - \boldsymbol{X_d}) + \boldsymbol{U}^T\boldsymbol{R}\boldsymbol{U} + \Delta\boldsymbol{U}^T\Delta\boldsymbol{R}\Delta\boldsymbol{U} \tag{59}$$

in which, $\Delta\boldsymbol{U}$ is the input increment vector and is defined as follows:

$$\Delta\boldsymbol{U} = \begin{bmatrix} \Delta\boldsymbol{u}(t) \\ \Delta\boldsymbol{u}(t+1) \\ \vdots \\ \Delta\boldsymbol{u}(t+N-1) \end{bmatrix} \tag{60}$$

In order to write the cost function as an explicit function of the input vector, $\Delta\boldsymbol{U}$ is recast as follows:

$$\Delta\boldsymbol{U} = \boldsymbol{S}\boldsymbol{U} - \bar{\boldsymbol{u}} \tag{61}$$

where:

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_8 & \boldsymbol{0}_{8\times 8} & \boldsymbol{0}_{8\times 8} & \cdots & \boldsymbol{0}_{8\times 8} \\ -\boldsymbol{I}_8 & \boldsymbol{I}_8 & \boldsymbol{0}_{8\times 8} & \cdots & \boldsymbol{0}_{8\times 8} \\ \boldsymbol{0}_{8\times 8} & -\boldsymbol{I}_8 & \boldsymbol{I}_8 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{0}_{8\times 8} \\ \boldsymbol{0}_{8\times 8} & \cdots & \boldsymbol{0}_{8\times 8} & -\boldsymbol{I}_8 & \boldsymbol{I}_8 \end{bmatrix}_{8N\times 8N} \tag{62}$$

$$\bar{\boldsymbol{u}} = [\boldsymbol{u}^T(t-1) \quad \boldsymbol{0}_{1\times 8} \quad \dots \quad \boldsymbol{0}_{1\times 8}]^T$$

It should be observed that the implementation of the input increment term necessitates the inclusion of the previously computed and applied control input in the last time step, $\boldsymbol{u}(t-1)$, within the control rule.

Accordingly, the terms $\boldsymbol{H}$ and $\boldsymbol{f}$ in equation (54) are given as follows:

$$\boldsymbol{H} = 2(\boldsymbol{\mathcal{B}}^T\boldsymbol{Q}\boldsymbol{\mathcal{B}} + \boldsymbol{R} + \boldsymbol{S}^T\Delta\boldsymbol{R}\boldsymbol{S}) \tag{63}$$

$$\boldsymbol{f} = 2(\boldsymbol{\mathcal{B}}^T\boldsymbol{Q}(\boldsymbol{\mathcal{A}}\boldsymbol{x}(t) - \boldsymbol{X_d}) - \boldsymbol{S}^T\Delta\boldsymbol{R}\bar{\boldsymbol{u}}) \tag{64}$$

3.3 Reaction Wheel Constraints

Reaction wheels are the most widely used actuators for satellite attitude control [11,12]. It is well-known that the angular momentum of reaction wheels are physically constrained, in addition to their produced torque. In order for a control law to be feasible, all physical constraints should be considered in the control rule. One of the major advantages of MPC is its capacity to explicitly involve constrains. The process of augmenting the previous designed control law with reaction wheels angular momentum constraint begins with discretization:

$$\boldsymbol{h}_w(t+1) = \boldsymbol{h}_w(t) + \Delta t \boldsymbol{\tau}(t) \tag{65}$$

In this relation, $\boldsymbol{h}_w$ is the angular momentum vector of the arbitrary set of reaction wheels in the body coordinates, and $\boldsymbol{\tau}$ is the control torque due to the variations of $\boldsymbol{h}_w$. In the simplest configuration, in which a single wheel is used for each body channel, each element of $\boldsymbol{h}_w$ is the non-zero element of the momentum vector of the corresponding wheel.

Accordingly, the future angular momentum vector in the prediction-control horizon is defined as follows:

$$\boldsymbol{H}_{w_{3N\times 1}} = \begin{bmatrix} \boldsymbol{h}_w(t+1) \\ \boldsymbol{h}_w(t+2) \\ \vdots \\ \boldsymbol{h}_w(t+N) \end{bmatrix} \tag{66}$$

Successive applying (65), the predictive model is obtained as follows:

$$\boldsymbol{H}_w = \begin{bmatrix} \boldsymbol{I}_3 \\ \boldsymbol{I}_3 \\ \vdots \\ \boldsymbol{I}_3 \end{bmatrix} \boldsymbol{h}_w(t) + \Delta t \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \dots & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{I}_3 & \boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} \\ \vdots \\ \boldsymbol{I}_3 & \boldsymbol{I}_3 & \dots & \boldsymbol{I}_3 & \boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}(t) \\ \boldsymbol{\tau}(t+1) \\ \vdots \\ \boldsymbol{\tau}(t+N-1) \end{bmatrix} \tag{67}$$

In order to fit with the dual framework, the above equation should be recast according to the dual input, $\boldsymbol{u}$. To this end, equation (40) is first expressed in reverse form:

$$\boldsymbol{\tau}(t) = \boldsymbol{F}\boldsymbol{u}(t) \tag{68}$$

where:

$$\boldsymbol{F} \triangleq [\boldsymbol{0}_{3\times 5} \quad \boldsymbol{I}_3]_{3\times 8} \tag{69}$$

Accordingly, $\boldsymbol{H}_w$ could be written as follows:

$$\boldsymbol{H}_w = \boldsymbol{\mathcal{I}}\boldsymbol{h}_w(t) + \Delta t \boldsymbol{\mathcal{F}}\boldsymbol{U} \tag{70}$$

in which:

$$\boldsymbol{\mathcal{F}}_{3N\times 8N} \triangleq \begin{bmatrix} \boldsymbol{F} & \boldsymbol{0} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{F} & \boldsymbol{F} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots \\ \boldsymbol{F} & \boldsymbol{F} & \dots & \boldsymbol{F} & \boldsymbol{F} \end{bmatrix} \tag{71}$$

$$\boldsymbol{\mathcal{I}}_{3N\times 3} \triangleq \begin{bmatrix} \boldsymbol{I}_3 \\ \boldsymbol{I}_3 \\ \vdots \\ \boldsymbol{I}_3 \end{bmatrix} \tag{72}$$

The bound constraint for $\boldsymbol{H}_w$ is naturally given as follows:

$$-\boldsymbol{H}_b \le \boldsymbol{H}_w \le \boldsymbol{H}_b \tag{73}$$

However, since the optimization variable in (57) is the future input vector, the above constraint should be rewritten as a constraint on $\boldsymbol{U}$:

$$\begin{aligned} &\boldsymbol{H}_w \le \boldsymbol{H}_b \\ &\Rightarrow \Delta t \boldsymbol{\mathcal{F}}\boldsymbol{U} \le \boldsymbol{H}_b - \boldsymbol{\mathcal{I}}\boldsymbol{h}_w(t) \\ &\boldsymbol{H}_w \ge -\boldsymbol{H}_b \\ &\Rightarrow -\Delta t \boldsymbol{\mathcal{F}}\boldsymbol{U} \le \boldsymbol{H}_b + \boldsymbol{\mathcal{I}}\boldsymbol{h}_w(t) \end{aligned} \tag{74}$$

Thus, the constraint is given in a compact and standard form as follows:

$$\Delta t \begin{bmatrix} \boldsymbol{\mathcal{F}} \\ -\boldsymbol{\mathcal{F}} \end{bmatrix} \boldsymbol{U} \le \begin{bmatrix} \boldsymbol{H}_b - \boldsymbol{\mathcal{I}}\boldsymbol{h}_w(t) \\ \boldsymbol{H}_b + \boldsymbol{\mathcal{I}}\boldsymbol{h}_w(t) \end{bmatrix} \tag{75}$$

3.4 Reaction Wheel Cost Term

The main restriction of using reaction wheels for attitude control is that their residual angular momentum eventually leads to saturation. The capacity of MPC could be exploited to reduce the momentum residual. This is achieved by penalizing the angular momentum vector magnitude of the wheels in the cost function. The quadratic cost term would be $\boldsymbol{H}_w^T \boldsymbol{R}_h \boldsymbol{H}_w$, in which $\boldsymbol{R}_h$ is the corresponding weight matrix. This term is rewritten in terms of $\boldsymbol{U}$, using (70), as follows:

$$\begin{aligned} \boldsymbol{H}_w^T \boldsymbol{R}_h \boldsymbol{H}_w = {} & \Delta t^2 \boldsymbol{U}^T \boldsymbol{\mathcal{F}}^T \boldsymbol{R}_h \boldsymbol{\mathcal{F}}\boldsymbol{U} \\ & + 2\Delta t \boldsymbol{h}_w^T(t) \boldsymbol{\mathcal{I}}^T \boldsymbol{R}_h \boldsymbol{\mathcal{F}}\boldsymbol{U} + \boldsymbol{\mathcal{C}} \end{aligned} \tag{76}$$

As before, $\boldsymbol{\mathcal{C}}$ represents the fix term. Accordingly, the improved cost function will be:

$$\begin{aligned} J = {} & (\boldsymbol{X} - \boldsymbol{X}_d)^T \boldsymbol{Q} (\boldsymbol{X} - \boldsymbol{X}_d) + \boldsymbol{U}^T \boldsymbol{R} \boldsymbol{U} \\ & + \Delta \boldsymbol{U}^T \Delta \boldsymbol{R} \Delta \boldsymbol{U} + \boldsymbol{H}_w^T \boldsymbol{R}_h \boldsymbol{H}_w \end{aligned} \tag{77}$$

which is recast in the form of (54) as follows:

$$\boldsymbol{H} = 2(\boldsymbol{\mathcal{B}}^T \boldsymbol{Q} \boldsymbol{\mathcal{B}} + \boldsymbol{R} + \boldsymbol{S}^T \Delta \boldsymbol{R} \boldsymbol{S} + \Delta t^2 \boldsymbol{\mathcal{F}}^T \boldsymbol{R}_h \boldsymbol{\mathcal{F}}) \tag{78}$$

$$\boldsymbol{f} = 2\big(\boldsymbol{\mathcal{B}}^T \boldsymbol{Q}(\boldsymbol{\mathcal{A}} \boldsymbol{x}(t) - \boldsymbol{X}_{\boldsymbol{d}}) - \boldsymbol{S}^T \Delta \boldsymbol{R} \overline{\boldsymbol{u}} + \Delta t \boldsymbol{\mathcal{F}}^T \boldsymbol{R}_h \boldsymbol{\mathcal{J}} \boldsymbol{h}_w(t)\big) \tag{79}$$

Recall that $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{J}}$ are defined in (71) and (72), respectively.

4. SIMULATION RESULTS

In this section, the performance of the proposed spacecraft 6-DOF relative motion control framework is assessed through simulations from multiple perspectives.

In the subsequent simulated scenarios, the follower spacecraft is the under-control agent which tends to control its roto-translational states relative to a leader spacecraft. The leader spacecraft flies in a circular 500 km altitude orbit whose orbital parameters are assumed to remain fixed with time. The initial orbital parameters of the follower is described in Table 1, in which subscripts L and 0 indicate the orbital parameters of the leader and the initial orbital parameters of the follower, respectively, and $r_\oplus$ is the Earth's average radius. The follower's initial orbit has been adjusted, with respect to the leader's orbit, such that the components of the relative position vector at the beginning of the scenario are in the order of kilometers.

Table 1. The initial orbital parameters of the follower.

Parameter	Description	Value
a_0	Semi-major axis	$r_\oplus + 500$ [km]
e_0	Eccentricity	$e_L + 0.001$
i_0	Inclination	$i_L + 0.1°$
ω_0	Arg. of perigee	ω_L
Ω_0	RAAN	Ω_L
θ_0	True anomaly	$\theta_L + 0.05°$

In terms of attitude, the body frame of the leader is assumed to be coincident with its own Local Vertical Local Horizontal (LVLH) frame throughout the simulation. The initial orientation of the follower's body frame with respect to its own LVLH frame, in the form of Euler angles in ZYX order, is considered as follows:

$$[\varphi_0 \quad \theta_0 \quad \psi_0] = [4 \quad -4 \quad -6] \text{ [deg]} \tag{80}$$

In this investigation, the zero vector is considered as the desired relative state. In other words, the aim is to roto-translationally coordinate the follower with the leader.

The follower is considered to be a 10 kg satellite with the following inertia matrix, in [kg·m^2]:

$$\boldsymbol{J} = \begin{bmatrix} 1.8140 & -0.1185 & 0.0275 \\ -0.1185 & 1.7350 & 0.0169 \\ 0.0275 & 0.0169 & 3.4320 \end{bmatrix} \tag{81}$$

Furthermore, the dominant roto-translational disturbances in Low Earth Orbits (LEO), including aerodynamic drag force and moment, J_2 gravitational effect and gravity gradient moment, have been taken into account.

The constraints of reaction wheels are set based on a typical product (an example of such a product is NRWA-T065, by NewSpace Systems). Specifically, it is considered that $|\boldsymbol{\tau}| \leq 0.123$ N.m and also $|\boldsymbol{h}_w| \leq 0.65$ N.m.s. Furthermore, the limit of the control force is defined as $|\boldsymbol{f}| \leq 10$ N. The cost function weight matrices are determined, based on the general principles of MPC design, as detailed in Table 2, in which $N = 10$ indicates the prediction horizon.

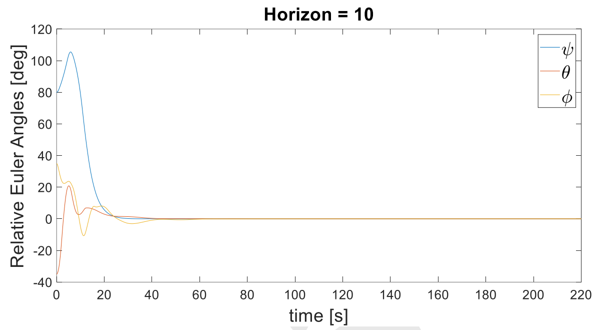
Table 2. The designed cost function weight matrices.

Parameter	Value
$\boldsymbol{Q}$	$\boldsymbol{I}_{16N}$
$\boldsymbol{R}$	$\boldsymbol{0}_{8N\times 8N}$
$\Delta\boldsymbol{R}$	$10^{-8} \times \boldsymbol{I}_{8N}$
$\boldsymbol{R}_h$	$10^{-1} \times \boldsymbol{I}_{3N}$

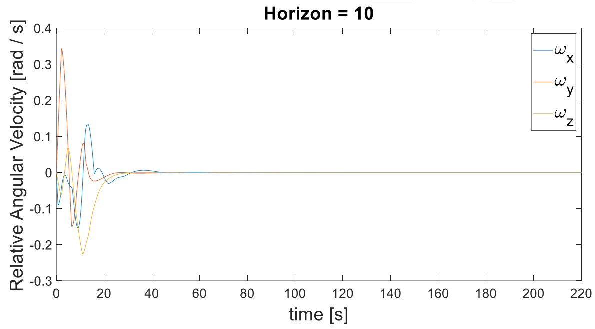
4.1 PWA-MPC with Reaction Wheels Constraints

First, the performance of the PWA-MPC including reaction wheels constraints (and no corresponding costs) is evaluated ($\boldsymbol{R}_h = \boldsymbol{0}$). The results are presented in Fig. 1.

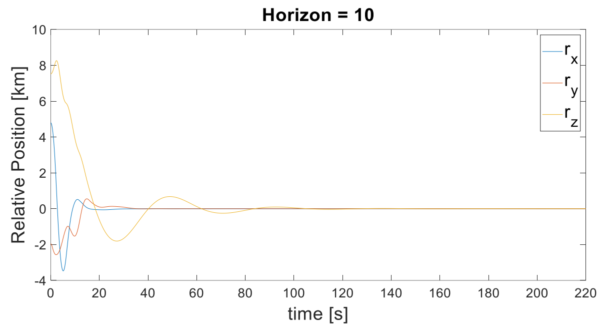
It can be clearly seen that the controller has brought the system to the desired roto-translational states with a fine agility, despite the severe disturbances and uncertainties. Furthermore, the force and moment constraints, as well as the constraints of the reaction wheels angular momentum, have been accurately satisfied. Also, the control effort has been vanished after the command is reached.



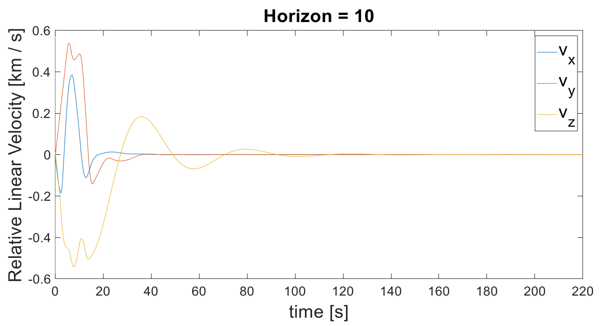
(A)



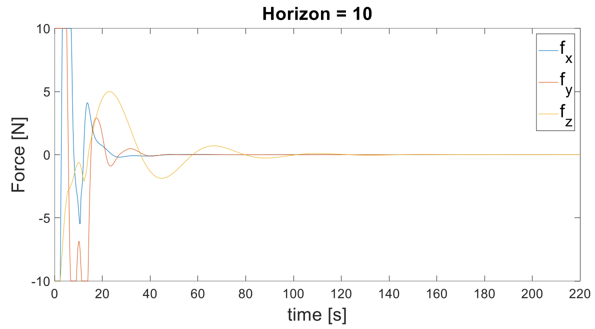
(B)



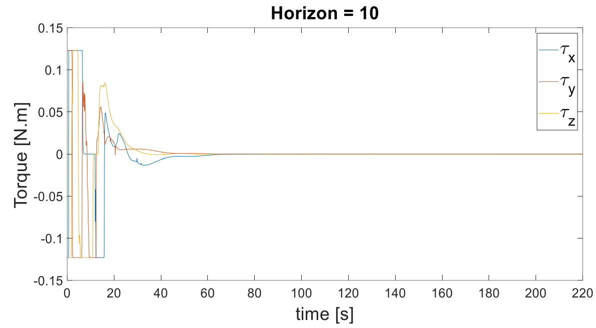
(C)



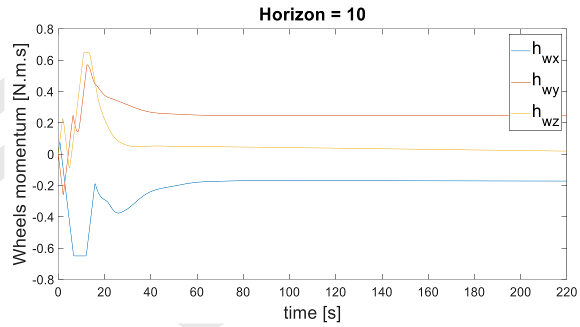
(D)



(E)



(F)



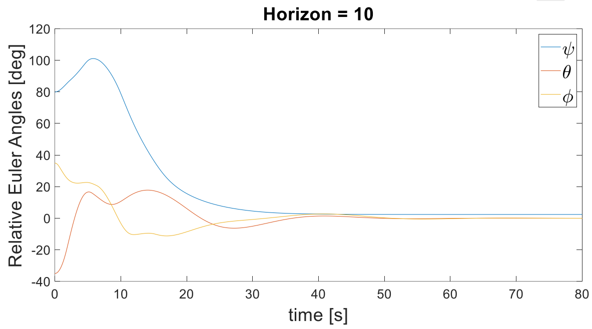
(G)

Fig. 1. Results for PWA-MPC with reaction wheels constraints (no costs): a) relative Euler angles, b) relative angular velocity, c) relative position, d) relative linear velocity, e) control force, f) control moment, g) angular momentum of reaction wheels.

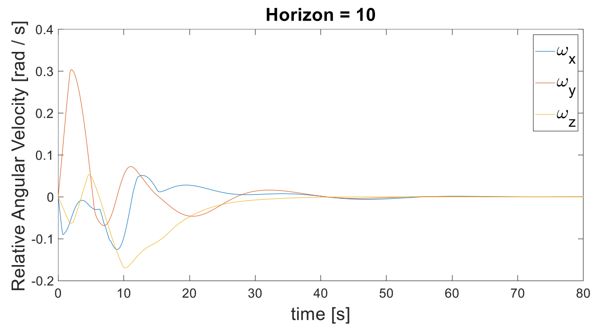
4.2 PWA-MPC with Reaction Wheels Constraints and Costs

Performance of the PWA-MPC including the constraint and penalty term due to the reaction wheels is studied in this section. The conditions of the simulation are exactly the same as previous scenario. The weight matrices of the controller are listed in Table 2 (including $\boldsymbol{R}_h$). The constraints are also adjusted as before. Thus, the results are precisely comparable.

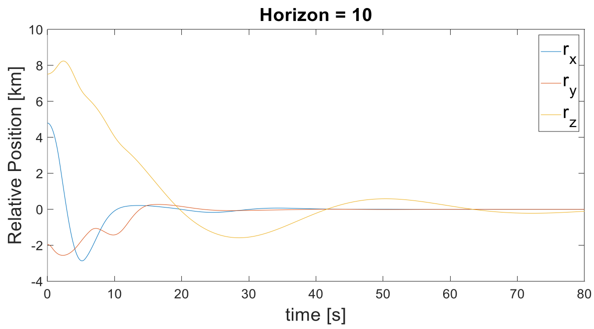
The results are provided in Fig. 2. It can be clearly seen that the constraints are again accurately satisfied. Furthermore, the ultimate absolute values of the wheels' momentum are considerably reduced. However, bias values appear in the ultimate Euler angles. This phenomenon arises as a consequence of the angular momentum penalty term associated with the wheels. In fact, this term is of the nature of time integral of control torque. It has been discussed in the relevant literature that inclusion of penalty for the input and its integral in the cost function leads to bias in the MPC output [13,14]. By compromising between the Euler angles steady state and the ultimate value of the wheels' angular momentum, the optimum weight matrices could be determined.



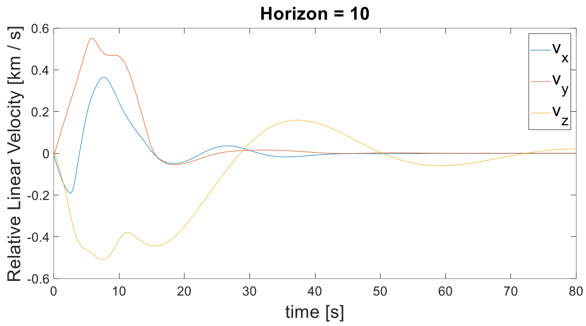
(A)



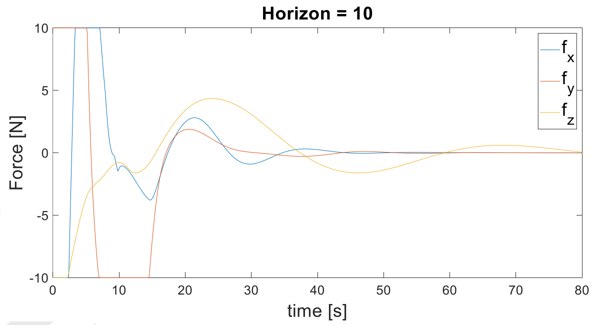
(B)



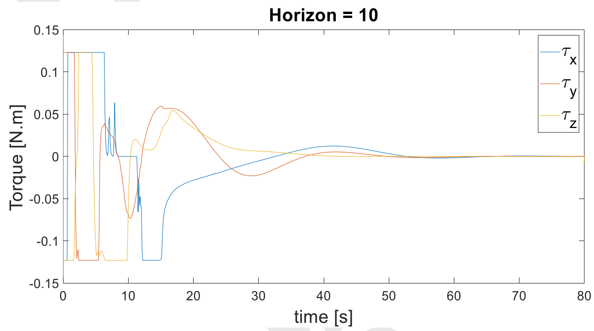
(C)



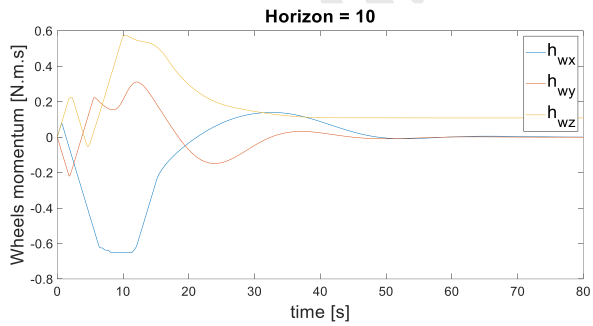
(D)



(E)



(F)



(G)

Fig. 2. Results for PWA-MPC with reaction wheels constraints and costs: a) relative Euler angles, b) relative angular velocity, c) relative position, d) relative linear velocity, e) control force, f) control moment, g) angular momentum of reaction wheels.

5. CONCLUSION

Potential complementary combination of MPC and dual-quaternion model of spacecraft roto-translational dynamics has been investigated. First, a short overview of dual-quaternion basics and tools is undertaken to illuminate the connection points. Subsequently, a piece-wise affine MPC framework is adopted based on a dual-quaternion model. This framework is then further developed for practical space missions involving roto-translational relative motion control. The spacecraft is assumed to include reaction wheels as attitude actuators. Simulation studies have demonstrated the effectiveness of the proposed framework.

The proposed framework paves the way for further research into combined roto-translational relative motion control of space agents in various aspects. Some subsequent research directions currently underway by the authors, with results expected to be published in the near future, include the following:

The proposed MPC framework is to be enhanced against uncertainties such as model-mismatches and external disturbances as well as actuator faults. Upon successful completion of the mentioned task, the validation level is to be upgraded via Hardware in the Loop (HIL) tests and experimental studies, whose theoretical foundations are presented in the current work.

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

REFERENCES

[1] X. Guo, C. Wang and L. Liu, “Adaptive fault-tolerant control for a class of nonlinear multi-agent systems with multiple unknown time-varying control directions,” *Automatica,* vol. 167, 2024, Art. no. 111802, https://doi.org/10.1016/j.automatica.2024.111802.

[2] R. Kinzie, R. Bevilacqua, D. Seo, J. W. Conklin and P. J. Wass, “Dual quaternion-based dynamics and control for gravity recovery missions,” *Acta Astronautica,* vol. 213, pp. 764-776, 2023, https://doi.org/10.1016/j.actaastro.2023.10.007.

[3] M. Nazari, E. A. Butcher, T. Yucelen and A. K. Sanyal, “Decentralized consensus control of a rigid-body spacecraft formation with communication delay,” *Journal of Guidance, Control, and Dynamics,* vol. 39, no. 4, pp. 838-851, 2016, https://doi.org/10.2514/1.G001396.

[4] D. Condurache, “On six DOF relative orbital motion of satellites,” in *Space Flight*, G. Dekoulis, Ed. IntechOpen, 2018, pp. 79-100, https://doi.org/10.5772/intechopen.73563.

[5] L. Sun and Z. Zheng, “Adaptive relative pose control of spacecraft with model couplings and uncertainties,” *Acta Astronautica,* vol. 143, pp. 29-36, 2018, https://doi.org/10.1016/j.actaastro.2017.11.006.

[6] S. V. Rakovi´c and W. S. Levine, *Handbook of Model Predictive Control*, Birkhauser Cham, 2019, https://doi.org/10.1007/978-3-319-77489-3.

[7] U. Lee and M. Mesbahi, “Constrained autonomous precision landing via dual quaternions and model predictive control,” *Journal of Guidance, Control, and Dynamics,* vol. 40, no. 2, p. 292–308, 2017, https://doi.org/10.2514/1.G001879.

[8] Q. Hu, H. Yang, H. Dong and X. Zhao, “Learning-based 6-DOF control for autonomous proximity operations under motion constraints,” *IEEE Transactions on Aerospace and Electronic Systems,* vol. 57, no. 6, pp. 4097-4109, 2021, https://doi.org/10.1109/TAES.2021.3094628.

[9] Q. Dang, H. Gui and H. Wen, “Dual-quaternion-based spacecraft pose tracking with a global exponential velocity observer,” *Journal of Guidance, Control, and Dynamics,* vol. 42, no. 9, pp. 2106-2115, 2019, https://doi.org/10.2514/1.G004302.

[10] E. Gallo, “The SO(3) and SE(3) lie algebras of rigid body rotations and motions and their application to Discrete Integration, Gradient Descent Optimization, and State Estimation,” arXiv:2205.12572, 2023, https://doi.org/10.48550/arXiv.2205.12572.

[11] S. A. Khan, Z. Tao, S. Fahad, M. Salman, M. Tahir and A. Ali, “Reliable attitude control integrating reaction wheels and embedded asymmetric magnetorquers for detumbling CubeSats,” *Advances in Space Research,* vol. 74, no. 11, pp. 5906-5922, 2024, https://doi.org/10.1016/j.asr.2024.08.056.

[12] G. Avanzini, E. d. Angelis, F. Giulietti and N. Serrano, “Attitude control of Low Earth Orbit satellites by reaction wheels and magnetic torquers,” *Acta Astronautica,* vol. 160, pp. 625-634, 2019, https://doi.org/10.1016/j.actaastro.2019.03.013.

[13] E. F. Camacho and C. Bordons, *Model Predictive Control*, 2nd Ed. London: Springer, 2007, https://doi.org/10.1007/978-0-85729-398-5.

[14] D. Xu, S. Damsma and M. Lazar, “On the steady-state behavior of finite-control-set MPC with an application to high-precision power amplifiers,” in *European Control Conference (ECC),* London, UK, 2022, https://doi.org/10.23919/ECC55457.2022.9838191.